New York Journal of Mathematics

Dancing polygons, rolling balls, and the Cartan-Engel distribution

Gil Bor and **Luis Hernández Lamoneda**

ABSTRACT. A pair of planar polygons is 'dancing' if one is inscribed in the other and they satisfy a certain cross-ratio relation at each vertex of the circumscribing polygon. Non-degenerate dancing pairs of closed n-gons exist for all $n \geq 6$. Dancing pairs correspond to trajectories of a non-holonomic mechanical system, consisting of a ball rolling, without slipping and twisting, along a polygon drawn on the surface of a ball 3 times larger than the rolling ball. The correspondence stems from reformulating both systems as piecewise rigid curves of a certain remarkable rank 2 non-integrable distribution defined on a 5-dimensional quadric in $\mathbb{RP}^6$, introduced by É. Cartan and F. Engel in 1893 in order to define the simple Lie group G_2.

CONTENTS

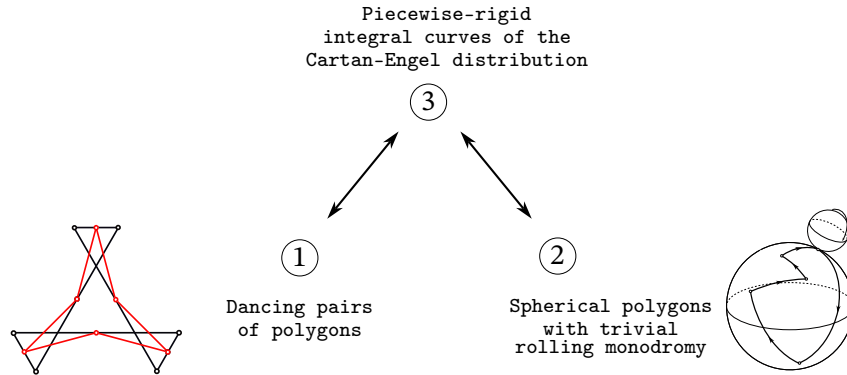## 1. Introduction and statement of results

This article connects two seemingly unrelated themes: (1) dancing pairs of polygons in the real projective plane, and (2) spherical polygons with trivial rolling monodromy. The connection is established by relating each of these themes to a third one: (3) piecewise-rigid integral curves of the Cartan-Engel distribution.

In this introductory section we describe themes (1) and (2) and state the relation between them, see Theorem 1.13. In the next section (Section 2) we describe theme (3) and the relations (1) ↔ (3) and (2) ↔ (3). Themes (2) and (3) and their relation were previously studied [3, 5, 7]; theme (1) and its relation to (3) is new and can be thought of as a discrete version of our previous work with P. Nurowski in [4]. The main new technical result of the paper, establishing the relation (1) ↔ (3), is Theorem 2.5 of Section 2 (proved in Section 3.1).

In Section 3, we give proofs of the theorems stated in the first two sections. In Appendix A, we give explicit formulas for the infinitesimal action of the group G_2 on the various configuration spaces appearing in this article (this action is due to a well-known symmetry property of the Cartan-Engel distribution, see Section 2.3 below). In Appendix B, we give explicit coordinate formulas for the 'rolling distribution' which models the rolling balls system of theme (2).

In general, we tried to make the article as self-contained as possible, without assuming the reader's familiarity with any of the mentioned themes.

1.1. Dancing pairs of polygons.

1.1. Dancing pairs of polygons. These are inscribed pairs of planar polygons, satisfying a certain system of scalar equations, one equation for each vertex of the circumscribing polygon, involving cross-ratios of neighboring vertices. The precise definition is as follows.

Consider a pair of polygons in the projective plane $\mathbb{RP}^2$, open or closed, the first with n vertices $A_1, A_2, \ldots, A_n$ and the second with n edges $b_1, b_2, \ldots, b_n, n \geq 2$. The second polygon is *inscribed* in the first if each vertex $B_i := b_i b_{i+1}$ (the intersection of b_i with b_{i+1}) lies on the edge $a_i := A_i A_{i+1}$ (the line through A_i and A_{i+1}). If the polygons are open then $i = 1, \ldots, n-1$ and if they are closed then $i = 1, \ldots, n$ and indices are considered mod n.
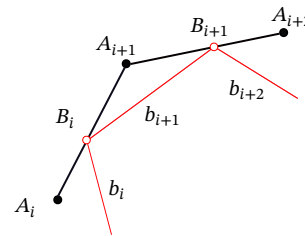
Definition 1.1 (Dancing pairs). An inscribed pair of polygons, as above, is a *dancing pair* if

$$[A_{i+1}, B_i, A_i, D] + [A_{i+1}, B_{i+1}, A_{i+2}, C] = 0, \tag{1}$$

where $C := b_i a_{i+1}$, $D := a_i b_{i+2}$, $i = 1, \dots, n-2$ for open polygons and $i = 1, \dots, n$ for closed polygons, in which case indices are considered mod n. See Figure 1.

Note that for $n = 2$ any inscribed (open) pair is automatically dancing.

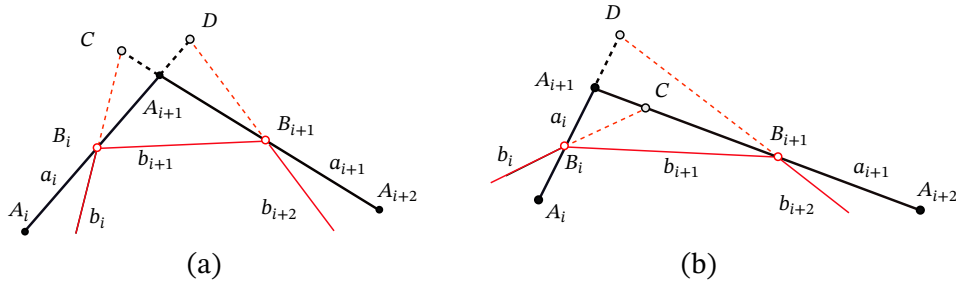(a)   (b)

FIGURE 1. The dancing condition, see Equation (1). Actually, (a) does not satisfy the dancing condition. Can you see why? See Remark 1.2.

The cross-ratio in Equation (1) is defined for 4 collinear points in $\mathbb{RP}^2$ by

$$[A_1, A_2, A_3, A_4] := \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}, \tag{2}$$

where x_i are the coordinates of A_i with respect to some affine coordinate along the containing line.

As is well known, the cross-ratio of 4 colinear points in $\mathbb{RP}^2$ is projectively invariant; that is, invariant under the action of the standard action of the projective group $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbb{RP}^2$.

Remark 1.2. Figure 1(a), although pleasantly symmetric and depicting correctly the definition of C, D in Equation (1), is not of a dancing pair since both summands on the left hand side of Equation (1) in this figure are positive. The reason is that, as can be easily verified, the cross-ratio $[A_1, A_2, A_3, A_4]$ is positive if and only if A_1, A_2 does not 'separate' A_3, A_4. See Figure 2.

$$[\bullet, \bullet, \circ, \circ] > 0 \qquad\qquad [\bullet, \bullet, \circ, \circ] < 0$$

FIGURE 2. The sign of the cross-ratio.

The term 'dancing' in Definition 1.1 is taken from [4]. The present article can be thought of as a discrete version of it.

For Definition 1.1 to make sense, one needs to make some genericity assumptions on the pair of polygons. Let us spell them out:

Definition 1.3. A pair of inscribed polygons in $\mathbb{RP}^2$, the first with vertices $A_1, \dots, A_n$ and the second with edges $b_1, \dots, b_n$, is *non-degenerate* if

- Each vertex A_i of the first polygon does not lie on the 'opposite' side b_i of the second polygon.
- Each three consecutive vertices of the first polygon are non-collinear.
- Each three consecutive edges of the second polygon are non-concurrent.

Existence of closed dancing pairs. It is easy to produce examples of non-degenerate dancing pairs of *open* polygons for all $n \geq 2$. For example, one picks arbitrary (generic) $A_1, \dots, A_n, b_1$, then arbitrary (generic) b_2 incident to $b_1(A_1 A_2)$, after which $b_3, \dots, b_n$ are determined recursively by equation (1) for $i = 1, \dots, n-2$. This gives a total of $(2n + 3)$-parameter family of open dancing pairs. When the polygons are closed, one needs to add Equation (1) also for $i = n-1, n$ and the equation $b_{n+1} = b_1$, reducing the number of parameters to $2n$. (One can also make this naive parameter count by considering that pairs of inscribed n-gons depend on $3n$ parameters and the n dancing conditions reduce this to $2n$.)

However, by examining the signs of the summands in Equation (1), one can see easily that there are no dancing pairs of triangles. More generally, we have the following result, which will be proved in later sections.

Theorem 1.4. *There are non-degenerate dancing pairs of closed n-gons if and only if $n \geq 6$.*

Figure 3 below shows an example of a non-degenerate dancing pair of closed hexagons.
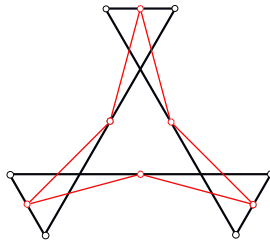
FIGURE 3. A dancing pair of hexagons.

Symmetries. Clearly, by the projective invariance of the cross-ratio appearing in Definition 1.1, the 8-dimensional projective group $\mathrm{PGL}_3(\mathbb{R})$ acts on the space of closed dancing pairs, so one expects, for n big enough, a $(2n - 8)$-parameter family of projective congruence classes (e.g., a 4-parameter family of non-trivial deformations of the example of Figure 3). A somewhat surprising construction in this paper (see Section 2.3) is an effective (local) action of the 14-dimensional

exceptional simple non-compact Lie group G_2 on the space of closed dancing pairs, so one expects a $2n - 14$ family of such congruence classes for $n \geq 8$. For $n = 6, 7$ one expects a discrete family of G_2-congruence classes.

Conjecture 1.5. *For $n = 6, 7$, all closed dancing pairs of n-gons are G_2 congruent.*

1.2. Spherical polygons, rolling balls and monodromy. The second theme of this article is a well-known non-holonomic mechanical system, see [1–3,5,7]. Consider a 'stationary' round sphere of radius r in Euclidean $\mathbb{R}^3$, on which a closed oriented polygonal path Γ is drawn (its edges are arcs of great circles). We impose a *non-degeneracy* condition on Γ: no three consecutive vertices are 'collinear', i.e., lie on the same great circle. Note that this implies that no two consecutive vertices are antipodal.

Next, take another sphere, a 'moving' sphere, of radius r', place it outside the stationary sphere, touching it at one of the vertices of Γ, then roll it along Γ without slipping or twisting (see Figure 4). A formal definition of these terms will be given later in Section 2.2.

As we roll the moving sphere along Γ, it rotates about its center. After going once around Γ, the moving sphere returns to the initial vertex, but possibly with a different orientation, given by an element of the orthogonal group SO_3, called the *rolling monodromy* of Γ. Put differently, as we roll the moving sphere along Γ, its rotation about its center defines a curve in SO_3, starting at the identity, whose other endpoint is the rolling monodromy of Γ. The curve in SO_3 can be lifted to a curve in the universal double-cover $S^3 \to SO_3$, starting at $1 \in S^3$ (we are thinking of S^3 as the sphere of unit quaternions), whose other end point is the *lifted rolling monodromy* of Γ.
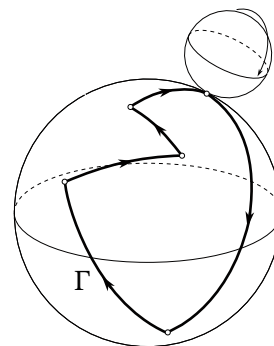
FIGURE 4.

Definition 1.6. The rolling monodromy of Γ is *trivial* if it is the identity element in SO_3. Similarly, for the lifted monodromy.

In other words, the rolling monodromy of Γ is trivial if the associated curve in SO_3 is *closed*, and the lifted monodromy is trivial if the lifted curve in S^3 is closed, which is the same as requiring the curve in SO_3 to be closed and *null homotopic*.

Clearly, if we pick another initial point on Γ then the (lifted) rolling monodromy differs by conjugation by an element in SO_3, which does not affect its triviality.

Note that the (lifted) rolling monodromy, and in particular its triviality, does depend on the radius ratio of the two spheres, $\rho := r/r'$.

Example 1.7. Let Γ be the equator of the stationary sphere (a horizontal great circle). Rolling the moving sphere once around Γ results in it being rotated

$\rho + 1$ times about the vertical axis through its center (this is elementary, but rather counterintuitive; see this animation [13]). Lifted to S^3, we get a path going $(\rho + 1)/2$ times around a great circle of S^3. Thus, the rolling monodromy of such a Γ is trivial if and only if ρ is an integer, and the lifted monodromy is trivial if and only if ρ is an *odd integer*.

Example 1.8. (We used this example to produce Figure 3.) Let Γ be an 'octant', i.e., an equilateral spherical triangle, with side length a quarter of a great circle. For $\rho = 3$, each of its sides, by the previous example, results in a lifted monodromy of $-1 \in S^3$, adding up to a total lifted monodromy of $(-1)^3 = -1$. Thus, rolling *twice* around Γ results in a trivial lifted monodromy.
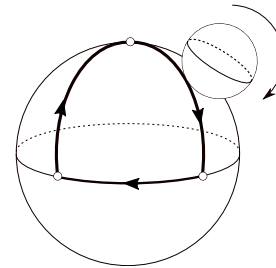
FIGURE 5.

We next present an infinite family of examples of non-degenerate regular spherical n-gons with trivial lifted rolling monodromy, valid for $\rho = 3$ and all $n \geq 6$.

Regular spherical polygons with trivial (lifted) monodromy. Consider a closed non-degenerate regular spherical n-gon Γ, contained in the (open) northern hemisphere of the stationary sphere, and whose vertices lie on a circle of latitude of radius $\phi \in (0, \pi/2)$ (the 'colatitude').

Let $w \in \mathbb{N}$ be the winding number of Γ about the north pole. Note that by the non-degeneracy assumption, Γ does not pass through the poles, so w is well defined. The 'rotation angle' between two successive vertices is then $\theta := 2\pi w/n$, where $0 < \theta < \pi$, i.e., $0 < w < n/2$. See Figure 6.

FIGURE 6. A regular spherical pentagon, projected onto the xy plane, with winding number $w = 2$ about the north pole O and rotation angle of $\theta = 4\pi/5$.

We ask: for which (n, w, ϕ, ρ) is the (lifted) rolling monodromy of Γ trivial? We shall answer this question only for $\rho = 3$, the case that interests us here.

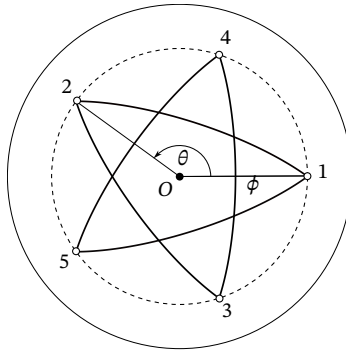**Theorem 1.9.** *Let Γ be a non-degenerate regular spherical n-gon on the stationary sphere, with winding number w about its center and circumscribing circle of radius ϕ. Furthermore, we assume that $n \geq 3$, $0 < w < n/2$ and $\phi \in (0, \pi/2)$, as described above. Then*

(a) Γ *has trivial rolling monodromy for $\rho = 3$ if and only if there exists a (necessarily unique) integer w' in the range $w < w' < n$ such that*

$$\cos\left(\frac{\pi w'}{n}\right) = \cos\left(\frac{\pi w}{n}\right)\left[1 - 4\sin^2\left(\frac{\pi w}{n}\right)\sin^2\phi\right]. \tag{3}$$

(b) *The lifted rolling monodromy of such Γ is trivial if and only if $w' \equiv w$ (mod 2).*

(c) *w' in Equation (3) is the winding number of the closed regular polygon traced on the moving sphere as it rolls along Γ.*

(d) *There are solutions to Equation (3) with $w' \equiv w$ (mod 2) if and only if $n \geq 6$. In fact, there is a solution for $w = 2$, $w' = 4$ and all $n \geq 6$.*

See Section 3.2 for a proof. Figure 7 shows some examples with ≤ 10 vertices.

$(6, 2, 4)$ $(7, 2, 4)$ $(8, 3, 5)$ $(9, 2, 4)$ $(10, 4, 6)$
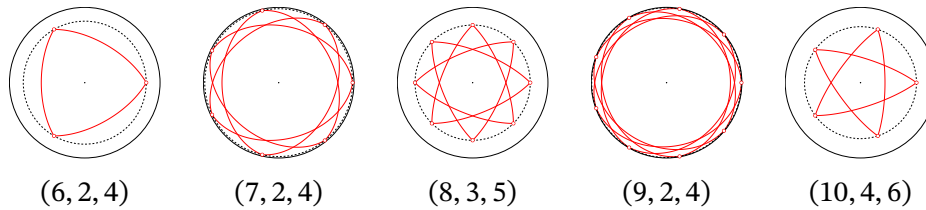
FIGURE 7. Regular spherical n-gon, $n = 6, 7, 8, 9, 10$, with trivial lifted rolling monodromy for $\rho = 3$, projected to the xy plane. The triple of numbers below each figure is (n, w, w').

Corollary 1.10. *For each $n \geq 6$ there exists a closed non-degenerate spherical n-gon with lifted trivial monodromy for $\rho = 3$.*

Remark 1.11. Theorem 1.9 does not give an explicit list of all regular polygons with trivial lifted monodromy for $\rho = 3$, since it only reduces the question to Equation (3), without fully solving it. We will not dwell on solving this equation completely, since the solutions given in part (d) of Theorem 1.9 are sufficient for this article. However, some numerical experiments indicate the following[1]:

- If (n, w, w') is an admissible triple (i.e., one can find $\phi \in (0, \pi/2)$ that solves the equation, with $0 < w < w' < n$, $w < n/2$ and $w \equiv w'$ mod 2) then so are all (n', w, w') with $n' > n$.

---

[1]We thank Carlos Licea, an undergraduate physics student of the University of Guanajuato, for helping with these experiments.

- Let us say that an admissible triple (n, w, w') is *minimal* if (n', w, w') is not admissible for $n' < n$. Then, for a fixed n, the minimal admissible triples and their number m are as follows, with $k \geq 2$ and $j = 1, 2, \ldots, m$ in all 3 cases:
    - If $n = 3k$ then $w = k + j - 1, w' = n - 3j + 1, m = [k/2]$.
    - if $n = 3k + 1$ then $w = k + j, w' = n - 3j - 1, m = [(k - 1)/2]$.
    - if $n = 3k + 2$ then $w = k + j, w' = n - 3j, m = [k/2]$.
- In particular, there are minimal admissible triples for all $n \geq 6$, except $n = 7$. The minimal triples for $n \leq 12$ are $(6, 2, 4)$, $(8, 3, 5)$, $(9, 3, 7)$, $(10, 4, 6)$, $(11, 4, 8)$, $(12, 4, 10)$, $(12, 5, 7)$.

1.3. The main theorem. Our main result relates dancing pairs of closed polygons with spherical polygons with lifted trivial rolling monodromy for radius ratio $\rho = 3$. What is special about $\rho = 3$ (as well as $1/3$) will be explained in the next section, once we interpret sphere rolling in terms of the Cartan-Engel distribution. For the moment, to state the relation, we need to add a definition.

Note that a non-degenerate spherical polygon Γ is not determined by its vertices: for each pair of successive vertices there are infinitely many 'edges' (directed arcs of great circles) connecting them; two of which are 'simple', i.e., non self-intersecting, complementary arcs of the great circle containing the two points, and the rest wrap around this circle an arbitrary number of times.

Definition 1.12. Two closed spherical polygons are *equivalent* if they have the same set of vertices, and/or some of the vertices are replaced by their antipodes.

In other words, equivalence classes of non-degenerate closed spherical n-gons are given by non-degenerate ordered sets of n points in $S^2 / \pm 1 \cong \mathbb{RP}^2$.

Note that, by Example 1.7, triviality of the lifted rolling monodromy of Γ for $\rho = 3$ is preserved by this equivalence relation (see Lemma 3.6 below for further details).

The initial placement of the moving sphere at one of the vertices of Γ is given by an element of SO_3, the orientation of the moving sphere with respect to some fixed reference orientation. The lifted path in S^3, describing the motion of the moving sphere along Γ, is thus determined by an arbitrary initial element $q \in S^3$ chosen 'above' the initial vertex in Γ.

Theorem 1.13 (main)**.** *There is a bijection between non-degenerate dancing pairs of closed n-gons in $\mathbb{RP}^2$ and generic[2] pairs $([\Gamma], q)$, where $q \in S^3$ and $[\Gamma]$ is an equivalence class of non-degenerate closed spherical n-gons with trivial lifted rolling monodromy for $\rho = 3$.*

For example, to the spherical octant of Figure 5, traversed twice, corresponds the dancing pair of hexagons of Figure 3. Similarly, the regular spherical polygons with trivial lifted monodromy of Theorem 1.9 correspond to examples of dancing pairs of n-gons for all $n \geq 6$ (one half of Theorem 1.4).

---

[2]The precise meaning of 'generic' will be given in the next section, once we describe the bijection in detail.

2. The Cartan-Engel distribution

The relation between dancing pairs and rolling balls, as indicated in Theorem 1.13, is based on modeling both problems by the same remarkable geometric object: a certain non-integrable rank 2 distribution on a 5-manifold, introduced by Elié Cartan and Friedrich Engel in 1893 (seemingly independently), in order to define the simple exceptional 14-dimensional Lie group G_2 [8, 10]. The subject has since been studied extensively by many authors (including ourselves, see [4, 5]). Here is a quick review of the basic properties relevant here.

Let $\mathscr{D}$ be a rank 2 distribution on a 5-manifold Q, i.e., a rank 2 subbundle of TQ. It is said to be a $(2, 3, 5)$-distribution if $[[\mathscr{D}, \mathscr{D}], \mathscr{D}] = TQ$; that is, for any local framing X_1, X_2 of $\mathscr{D}$ (i.e., two everywhere independent local sections of $\mathscr{D}$), let $X_3 := [X_1, X_2], X_4 := [X_1, X_3], X_5 := [X_2, X_3]$, then $X_1, \dots, X_5$ is a local framing of TQ. In fact, a generic rank 2 distibution on a 5-manifold is $(2, 3, 5)$ but typically no two are diffeomorphic, even locally. A (local) symmetry of a distribution $\mathscr{D}$ on a manifold Q is a (local) self-diffeomorphism of Q preserving $\mathscr{D}$.

In his fundamental paper on the subject [9], Cartan showed that the maximal dimension of the local symmetry group of a $(2, 3, 5)$-distribution is 14, in which case the local symmetry group is the simple non-compact Lie group G_2 and the distribution is called *flat*. Cartan also showed in [9] that all flat $(2, 3, 5)$-distributions are locally diffeomorphic. Cartan and Engel gave in 1893 explicit formulas for such a distribution on $\mathbb{R}^5$ (their formulas in [8, 10] look similar to our formula of Equation (6) below). We thus call a flat $(2, 3, 5)$-distribution a *Cartan-Engel distribution*, or, by a slight abuse of terminology, *the* Cartan-Engel distribution.

Another result of Cartan in [9] is *Cartan's submaximal symmetry statement*: the local symmetry group of a non-flat $(2, 3, 5)$-distribution is at most 7 dimensional.

A much more recent general result on $(2, 3, 5)$-distributions, by Bryant and Hsu [7] (see also the last paragraph of [6]), is the existence of *rigid* (or 'abnormal') integral curves: any small enough segment of such a curve admits no deformations within the class of integral curves with the same end-points. In fact, the rigid curves of a $(2, 3, 5)$-distribution $\mathscr{D}$ form (locally) a 5-parameter family of $\mathscr{D}$-integral curves, a unique such curve passing through a given point in Q in a given direction at the point tangent to $\mathscr{D}$.

We next describe briefly 3 models of the Cartan-Engel distribution that are used in this paper and their interrelations.

2.1. First model: dancing pairs. Let $\mathbb{R}^{3,3} := \mathbb{R}^3 \times (\mathbb{R}^3)^*$, a 6-dimensional real vector space equipped with the quadratic form $(\mathbf{A}, \mathbf{b}) \mapsto \mathbf{bA}$ of signature $(3, 3)$ (we are thinking of $\mathbb{R}^3$ as column vectors and $(\mathbb{R}^3)^*$ as row vectors). We use the standard volume forms vol, vol^* on $\mathbb{R}^3, (\mathbb{R}^3)^*$ (respectively) to define 'vector products' $\mathbb{R}^3 \times \mathbb{R}^3 \to (\mathbb{R}^3)^*, (\mathbb{R}^3)^* \times (\mathbb{R}^3)^* \to \mathbb{R}^3$ by

$$\mathbf{A}_1 \times \mathbf{A}_2 := vol(\mathbf{A}_1, \mathbf{A}_2, \cdot), \quad \mathbf{b}_1 \times \mathbf{b}_2 = vol^*(\mathbf{b}_1, \mathbf{b}_2, \cdot). \tag{4}$$

Denote the projection $\mathbb{R}^3 \setminus 0 \to \mathbb{RP}^2$ by $\mathbf{A} \mapsto [\mathbf{A}]$, and similarly for $(\mathbb{R}^3)^*$. That is, we are thinking of $\mathbb{R}^3$ and $(\mathbb{R}^3)^*$ as homogeneous coordinates of points and lines in $\mathbb{RP}^2$ (respectively).

Next, let $Q^{\mathrm{dan}} \subset \mathbb{R}^{3,3}$ be the 5-dimensional affine quadric

$$Q^{\mathrm{dan}} := \{(\mathbf{A}, \mathbf{b}) \mid \mathbf{b}\mathbf{A} = 1\}.$$

Clearly, the tangent space to Q^{dan} at a point $(\mathbf{A}, \mathbf{b})$, translated to the origin, consists of vectors $(\dot{\mathbf{A}}, \dot{\mathbf{b}}) \in \mathbb{R}^{3,3}$ satisfying

$$\dot{\mathbf{b}}\mathbf{A} + \mathbf{b}\dot{\mathbf{A}} = 0 \tag{5}$$

Definition 2.1. Define a rank 2 distribution $\mathscr{D}^{\mathrm{dan}} \subset TQ^{\mathrm{dan}}$ as follows: its elements at a point $(\mathbf{A}, \mathbf{b}) \in Q^{\mathrm{dan}} \subset \mathbb{R}^{3,3}$, translated to the origin in $\mathbb{R}^{3,3}$, are the vectors $(\dot{\mathbf{A}}, \dot{\mathbf{b}}) \in \mathbb{R}^{3,3}$ satisfying, in addition to Equation (5),

$$\dot{\mathbf{b}} = \mathbf{A} \times \dot{\mathbf{A}}. \tag{6}$$

Proposition 2.2. $(Q^{\mathrm{dan}}, \mathscr{D}^{\mathrm{dan}})$ *is a flat* $(2, 3, 5)$*-distribution.*

This appeared in [4]. We recall the flatness argument: $(Q^{\mathrm{dan}}, \mathscr{D}^{\mathrm{dan}})$ admits $\mathrm{SL}_3(\mathbb{R})$ as an 'obvious' symmetry group, the restriction to Q^{dan} of the diagonal action on $\mathbb{R}^{3,3}$, $(\mathbf{A}, \mathbf{b}) \mapsto (g\mathbf{A}, \mathbf{b}g^{-1})$. This is an 8-dimensional group, hence by Cartan's submaximality result, the full symmetry group is in fact 14-dimensional.

An explicit local action of G_2 on Q^{dan} is also given in [4]. We recall this in Appendix A.2 by embedding $(Q^{\mathrm{dan}}, \mathscr{D}^{\mathrm{dan}})$ in another model of the Cartan-Engel distribution, a homogeneous space of G_2, constructed using split octonions. See Section 3.3.

Here is an easy to verify extrinsic property of $\mathscr{D}^{\mathrm{dan}}$, not found in [4].

Proposition 2.3. *For any given point in Q^{dan} and a $\mathscr{D}^{\mathrm{dan}}$-direction at the point, the affine line in $\mathbb{R}^{3,3}$ passing through the given point and tangent to the given direction is contained in Q^{dan} and everywhere tangent to $\mathscr{D}^{\mathrm{dan}}$. In fact, these affine lines are exactly the rigid curves of $\mathscr{D}^{\mathrm{dan}}$ in the sense of [7].*

Now, we come to the second main definition of this article (the first one was Definition 1.1).

Definition 2.4 (Horizontal polygons)**.** A *horizontal polygon* in Q^{dan} is a polygon in $\mathbb{R}^{3,3}$ whose edges are $\mathscr{D}^{\mathrm{dan}}$-horizontal lines in Q^{dan}, as in Proposition 2.3. The polygon is *non-degenerate* if every 3 consecutive vertices are non-collinear.

Our next theorem gives a bijection between the objects defined in Definitions 1.1 and 2.4.

Theorem 2.5. *For every non-degenerate horizontal n-gon in Q^{dan}, closed or open, with vertices $(\mathbf{A}_1, \mathbf{b}_1), (\mathbf{A}_2, \mathbf{b}_2), \ldots, (\mathbf{A}_n, \mathbf{b}_n)$, $n \geq 2$, the projected pair of polygons, the first with vertices $A_i = [\mathbf{A}_i] \in \mathbb{RP}^2$ and the second with edges $b_i = [\mathbf{b}_i] \in (\mathbb{RP}^2)^*$, $i = 1, \ldots, n$, is a non-degenerate dancing pair. Conversely, every non-degenerate dancing pair of polygons lifts uniquely to a non-degenerate horizontal polygon in Q^{dan}.*

This is proved in Section 3.1.

2.2. Second model: rolling balls.

The configuration space for rolling balls is $Q^{\texttt{roll}} := S^2 \times SO_3$. A point $(\mathbf{v}, g) \in Q^{\texttt{roll}}$ represents a placement of the moving ball so that it touches the stationary sphere at the point $r\mathbf{v}$ and is rotated with respect to some fixed reference orientation by $g \in SO_3$. Let $\varphi_{(\mathbf{v},g)} : \mathbb{R}^3 \to \mathbb{R}^3$ be the rigid motion $\mathbf{x} \mapsto g\mathbf{x} + (r + r')\mathbf{v}$. Then the placement of the moving ball is given by its image under $\varphi_{(\mathbf{v},g)}$. See Figure 8.
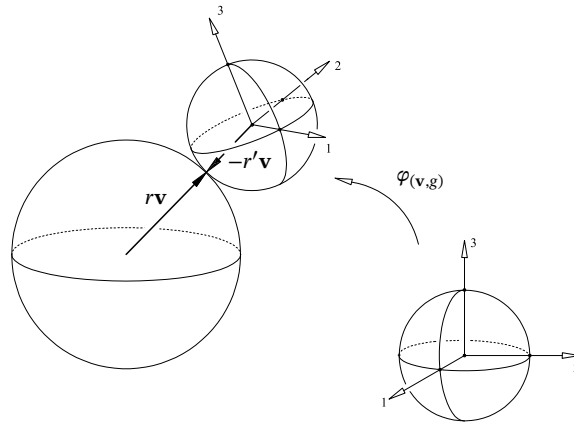

FIGURE 8. The rolling configuration space $S^2 \times SO_3$.

For a given element $g \in SO_3$ and tangent vector $\dot{g} \in T_g SO_3$ there is a unique $\boldsymbol{\omega} \in \mathbb{R}^3$, called the *angular velocity* vector, such that $\dot{g}g^{-1} \in \mathfrak{so}_3$ is given by $\mathbf{v} \mapsto \boldsymbol{\omega} \times \mathbf{v}$.

Definition 2.6. Define $\mathscr{D}^{\texttt{roll}} \subset TQ^{\texttt{roll}}$ as follows. The elements of $\mathscr{D}^{\texttt{roll}}$ at a point $(\mathbf{v}, g) \in Q^{\texttt{roll}}$ are the tangent vectors $(\dot{\mathbf{v}}, \dot{g}) \in T_{(\mathbf{v},g)}Q^{\texttt{roll}}$ satisfying

(1) $(\rho + 1)\dot{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{v}$,
(2) $\boldsymbol{\omega} \cdot \mathbf{v} = 0$.

See Proposition 1 of [5] for a derivation of conditions (1) and (2) from 'no-slip' and 'no-twist' (respectively).

Proposition 2.7.

(a) $\mathscr{D}^{\texttt{roll}}$ is a 2-distribution on $Q^{\texttt{roll}}$, integrable for $\rho = 1$ and $(2, 3, 5)$ for $\rho \neq 1$.
(b) For $\rho \neq 1$, the rigid curves of $\mathscr{D}^{\texttt{roll}}$ correspond to rolling along great circles of the stationary sphere. Thus rolling along polygonal curves on the stationary sphere corresponds to piecewise rigid curves of $\mathscr{D}^{\texttt{roll}}$.
(c) $\mathscr{D}^{\texttt{roll}}$ is a flat $(2, 3, 5)$-distribution if and only if $\rho = 3$ or $1/3$.

Parts (a) and (b) appeared in §4.4 of [7]. We learned about part (c) from a conversation with Robert Bryant. See [3, 5] for a proof of (c) as well as attempts to explain the mysterious 3:1 radius ratio.

2.3. Third model: split octonions. We use here the notation of [4, §3]. See also [11, 14] for more details.

Split octonions. This is an 8-dimensional non-commutative and non-associative real algebra $\mathbb{O}$. Following Zorn [16, page 144], its elements can be written as 'vector matrices'

$$\begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & y \end{pmatrix}, \ \ x, y \in \mathbb{R}, \ \ \mathbf{A} \in \mathbb{R}^3, \ \ \mathbf{b} \in (\mathbb{R}^3)^*,$$

with 'vector-matrix-multiplication',

$$\begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & y \end{pmatrix} \begin{pmatrix} x' & \mathbf{A}' \\ \mathbf{b}' & y' \end{pmatrix} := \begin{pmatrix} xx' - \mathbf{b}'\mathbf{A} & x\mathbf{A}' + y'\mathbf{A} + \mathbf{b} \times \mathbf{b}' \\ x'\mathbf{b} + y\mathbf{b}' + \mathbf{A} \times \mathbf{A}' & yy' - \mathbf{b}\mathbf{A}' \end{pmatrix},$$

where, as before, we use the 'vector products' of formulas (4).

Conjugation in $\mathbb{O}$ is given by

$$\zeta = \begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & y \end{pmatrix} \mapsto \overline{\zeta} = \begin{pmatrix} y & -\mathbf{A} \\ -\mathbf{b} & x \end{pmatrix},$$

satisfying

$$\overline{\overline{\zeta}} = \zeta, \ \ \overline{\zeta\zeta'} = \overline{\zeta'}\,\overline{\zeta}, \ \ \zeta\overline{\zeta} = \langle \zeta, \zeta \rangle \mathbf{1},$$

where $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the mutiplicative unit and

$$\langle \zeta, \zeta \rangle := xy + \mathbf{b}\mathbf{A} \tag{7}$$

is a quadratic form of signature $(4, 4)$ on $\mathbb{O}$.

Define as usual

$$\mathrm{Re}(\zeta) = (\zeta + \overline{\zeta})/2, \ \ \mathrm{Im}(\zeta) = (\zeta - \overline{\zeta})/2,$$

so that

$$\mathbb{O} = \mathrm{Re}(\mathbb{O}) \oplus \mathrm{Im}(\mathbb{O}),$$

where $\mathrm{Re}(\mathbb{O}) = \mathbb{R}\mathbf{1}$ and $\mathrm{Im}(\mathbb{O})$ are 'traceless' vector matrices of the form $\begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{pmatrix}$.

Let $\mathbb{RP}^6 = \mathbb{P}(\mathrm{Im}(\mathbb{O})) = (\mathrm{Im}(\mathbb{O}) \setminus 0) / \mathbb{R}^*$ (projectivized imaginary split octonions) and $Q^{\mathrm{oct}} \subset \mathbb{RP}^6$ the projectivized null cone; that is,

$$Q^{\mathrm{oct}} = \{[\zeta] \mid \zeta \in \mathrm{Im}(\mathbb{O}), \ \zeta \neq 0, \ \langle \zeta, \zeta \rangle = 0\}.$$

Definition 2.8. $\mathscr{D}^{\mathrm{oct}} \subset TQ^{\mathrm{oct}}$ is the distribution whose elements at a point $[\zeta] \in Q^{\mathrm{oct}}$ are given by the *annihilator*

$$\zeta^0 = \{\zeta' \in \mathrm{Im}(\mathbb{O}) \mid \zeta\zeta' = 0\},$$

as follows. For a non-zero null $\zeta \in \mathrm{Im}(\mathbb{O})$, ζ^0 is a 3-dimensional subspace of $\mathrm{Im}(\mathbb{O})$, containing $\mathbb{R}\zeta$ and tangent to the null cone at ζ, so descends to a projective 2-plane, tangent to Q^{oct} at $[\zeta]$. Its tangent space at $[\zeta]$ is the fiber of $\mathscr{D}^{\mathrm{oct}}$ at $[\zeta]$.

We have a projective analog of Proposition 2.3 and Definition 2.4.

Proposition 2.9. *For any given point in Q^{oct} and a $\mathscr{D}^{\mathrm{oct}}$-direction at the point, the projective line in $\mathbb{RP}^6$ passing through the given point and tangent to the given direction is contained in Q^{oct} and everywhere tangent to $\mathscr{D}^{\mathrm{oct}}$. These are the rigid curves of $\mathscr{D}^{\mathrm{oct}}$.*

Definition 2.10 (Horizontal polygons)**.** A *horizontal polygon* in Q^{oct} is a polygon in $\mathbb{RP}^6$ whose edges are $\mathscr{D}^{\mathrm{oct}}$-horizontal projective lines in Q^{oct}, as in Proposition 2.9. The polygon is *non-degenerate* if every 3 consecutive vertices are non-collinear.

G_2-*symmetry.* The automorphism group of $\mathbb{O}$, i.e., the subgroup of $\mathrm{GL}(\mathbb{O})$ preserving octonion multiplication, is a non-compact 14-dimensional connected simple Lie group, denoted by G_2. See Appendix A. The G_2-action on $\mathbb{O}$ preserves the splitting $\mathbb{O} = \mathrm{Re}(\mathbb{O}) \oplus \mathrm{Im}(\mathbb{O})$ and acts trivially on $\mathrm{Re}(\mathbb{O})$. The action on $\mathrm{Im}(\mathbb{O})$ induces an effective action on $\mathbb{P}(\mathrm{Im}(\mathbb{O}))$, clearly preserving $(Q^{\mathrm{oct}}, \mathscr{D}^{\mathrm{oct}})$. Therefore, one has

Proposition 2.11. $(Q^{\mathrm{oct}}, \mathscr{D}^{\mathrm{oct}})$ *is a flat* $(2, 3, 5)$-*distribution, whose symmetry group is* G_2.

The Lie algebra of G_2 is $\mathfrak{g}_2 \subset \mathfrak{so}_{4,3} \subset \mathfrak{so}_{4,4}$; the first inclusion is due to the G_2-invariance of the inner product of equation (7), restricted to $\mathrm{Im}(\mathbb{O})$, the second from the inclusion $\mathrm{Im}(\mathbb{O}) \subset \mathbb{O}$.

2.4. Putting it all together. To prove Theorem 1.13, we combine the three models of the Cartan-Engel distribution presented so far, as summed up in the following diagram.

$$
\begin{array}{ccc}
\widetilde{Q}^{\mathrm{oct}} & \xleftarrow[\sim]{\Phi} & \widetilde{Q}^{\mathrm{roll}} \\
\downarrow & & \downarrow \\
Q^{\mathrm{dan}} \overset{\iota}{\hookrightarrow} Q^{\mathrm{oct}} & & Q^{\mathrm{roll}}
\end{array}
\tag{8}
$$

In this diagram appear the underlying manifolds of our three models for the Cartan-Engel distribution, Q^{dan}, Q^{oct} and Q^{roll}, the universal covers of the last two (both of which are diffeomorphic to $S^2 \times S^3$), as well as the following maps:

- An embedding $\iota : Q^{\mathrm{dan}} \hookrightarrow Q^{\mathrm{oct}}$. One defines first an affine chart $\mathbb{R}^{3,3} \hookrightarrow \mathbb{P}(\mathrm{Im}(\mathbb{O})) = \mathbb{RP}^6$,

$$
(\mathbf{A}, \mathbf{b}) \mapsto [\zeta], \text{ where } \zeta = \begin{pmatrix} 1 & \mathbf{A} \\ \mathbf{b} & -1 \end{pmatrix},
$$

  then restrict to $Q^{\mathrm{dan}} = \{\mathbf{b}\mathbf{A} = 1\} \subset \mathbb{R}^{3,3}$. The image $\iota(Q^{\mathrm{dan}}) \subset Q^{\mathrm{oct}}$ is the complement in Q^{oct} of the 'hyperplane section' $\{\left[\begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{pmatrix}\right] \mid x = \mathbf{b}\mathbf{A} = 0\}$.
- The double covers $\widetilde{Q}^{\mathrm{oct}} \to Q^{\mathrm{oct}}$, $\widetilde{Q}^{\mathrm{roll}} \to Q^{\mathrm{roll}}$ (the vertical arrows).

– Recall that $Q^{\mathtt{oct}}$ is the projectivized null cone in

$$\mathbb{RP}^6 = (\mathrm{Im}(\mathbb{O}) \setminus 0) / \mathbb{R}^*.$$

If we quotient by $\mathbb{R}^+$ instead of $\mathbb{R}^*$ we get the 'spherized' null cone $\widetilde{Q}^{\mathtt{oct}} \subset (\mathrm{Im}(\mathbb{O}) \setminus 0) / \mathbb{R}^+ = S^6$, and a double cover $\widetilde{Q}^{\mathtt{oct}} \to Q^{\mathtt{oct}}$, $\mathbb{R}^+ \zeta \mapsto \mathbb{R}^* \zeta$. We use this double cover to pull back $\mathscr{D}^{\mathtt{oct}}$ to $\widetilde{\mathscr{D}}^{\mathtt{oct}}$ on $\widetilde{Q}^{\mathtt{oct}}$.

– Recall that $Q^{\mathtt{roll}} = S^2 \times \mathrm{SO}_3$, define $\widetilde{Q}^{\mathtt{roll}} := S^2 \times S^3$ and use the usual double cover $S^3 \to \mathrm{SO}_3$ to define the double cover $\widetilde{Q}^{\mathtt{roll}} \to Q^{\mathtt{roll}}$. Explicitly, to an element $q \in S^3 \subset \mathbb{R}^4 = \mathbb{H}$ one associates the orthogonal transformation of $\mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$, $\mathbf{v} \mapsto q\mathbf{v}\bar{q}$. Again, use this double cover to pull back $\mathscr{D}^{\mathtt{roll}}$ to $\widetilde{\mathscr{D}}^{\mathtt{roll}}$ on $\widetilde{Q}^{\mathtt{roll}}$.

• A diffeomorphism[3] $\Phi : \widetilde{Q}^{\mathtt{roll}} \to \widetilde{Q}^{\mathtt{oct}}$. For $(\mathbf{v}, q) \in \widetilde{Q}^{\mathtt{roll}} = S^2 \times S^3 \subset \mathrm{Im}(\mathbb{H}) \times \mathbb{H}$ one defines[4]

$$\Phi(\mathbf{v}, q) = \mathbb{R}^+ \begin{pmatrix} \mathrm{Re}(\mathbf{v}q) & \mathbf{v} + \mathrm{Im}(\mathbf{v}q) \\ \mathbf{v} - \mathrm{Im}(\mathbf{v}q) & -\mathrm{Re}(\mathbf{v}q) \end{pmatrix}. \tag{9}$$

Theorem 2.12. *In diagram* (8), Φ *is a diffeomorphism and all maps preserve the Cartan-Engel distributions on the respective spaces.*

This is proved in Section 3.3. Now we can give a more precise statement of our main theorem (Theorem 1.13). Let $Q^{\mathtt{oct}}_* = \iota(Q^{\mathtt{dan}}) \subset Q^{\mathtt{oct}}$ (the complement of the hyperplane section $x = 0$), and similarly $\widetilde{Q}^{\mathtt{oct}}_* \subset \widetilde{Q}^{\mathtt{oct}}, \widetilde{Q}^{\mathtt{roll}}_* \subset \widetilde{Q}^{\mathtt{roll}}, Q^{\mathtt{roll}}_* \subset Q^{\mathtt{roll}}$ the corresponding submanifolds under the maps of diagram (8). A polygonal horizontal path in $Q^{\mathtt{oct}}$ (respectively $\widetilde{Q}^{\mathtt{oct}}, \widetilde{Q}^{\mathtt{roll}}, Q^{\mathtt{roll}}$) is *generic* if all its vertices lie in $Q^{\mathtt{oct}}_*$ (respectively $\widetilde{Q}^{\mathtt{oct}}_*, \widetilde{Q}^{\mathtt{roll}}_*, Q^{\mathtt{roll}}_*$). A pair (Γ, q), where Γ is a closed non-degenerate spherical n-gon with vertices $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $q \in S^3$, is *generic* if the horizontal lift of Γ to $\widetilde{Q}^{\mathtt{roll}} = S^2 \times S^3$, starting at $(\mathbf{v}_1, q)$, is a generic horizontal polygon.

Corollary (Theorem 1.13). *Diagram* (8) *defines a bijective correspondence between non-degenerate dancing pairs of closed n-gons in $\mathbb{RP}^2$ and generic pairs $([\Gamma], q)$, where $q \in S^3$ and $[\Gamma]$ is an equivalence class of non-degenerate closed spherical n-gon with trivial lifted rolling monodromy for $\rho = 3$.*

In section 3.4, we shall prove Theorem 1.13 using Theorem 2.12.

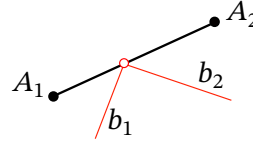# 3. Proofs

3.1. Theorem 2.5. Using induction on $n \geq 2$, the proof reduces to the following two lemmas, the $n = 2, 3$ cases of the theorem.

---

[3]This diffeomorphism is essentially the same as the one constructed in [3, Proposition 2].

[4]Note that for this formula to make sense we identify $\mathbb{R}^3$ with $(\mathbb{R}^3)^*$ using the standard Euclidean inner product.

Lemma 3.1 (Theorem 2.5 for $n = 2$). *Consider a horizontal non-degenerate 2-gon in Q^{dan}, given by a pair of vertices $q_1, q_2 \in Q^{\text{dan}}$ such that the line $q_1 q_2$ is horizontal. Then the projected pair of 2-gons, with vertices (A_1, A_2) and edges (b_1, b_2), is inscribed, i.e., $b_1 b_2 \in A_1 A_2$. Conversely, every inscribed pair of 2-gons lifts to a unique horizontal 2-gon in Q^{dan}.*

Proof. Suppose $q_i = (\mathbf{A}_i, \mathbf{b}_i)$, $i = 1, 2$, such that $q_1 q_2$ is horizontal. Then, by definition (Proposition 2.2), $\mathbf{A}_1 \times \mathbf{A}_2 = \mathbf{b}_2 - \mathbf{b}_1$. This equation implies $(\mathbf{A}_1 \times \mathbf{A}_2)(\mathbf{b}_1 \times \mathbf{b}_2) = 0$, i.e., $b_1 b_2 \in A_1 A_2$.

Conversely, suppose an inscribed pair of 2-gons is given, i.e., $b_1 b_2 \in A_1 A_2$, with $b_1 \neq b_2, A_1 \neq A_2, A_i \notin b_i, i = 1, 2$. The horizontality conditions on a lift $(\mathbf{A}_i, \mathbf{b}_i) \in Q^{\text{dan}}$, $i = 1, 2$, is $\mathbf{A}_1 \times \mathbf{A}_2 = \mathbf{b}_2 - \mathbf{b}_1$ (see Proposition 2.2).

Picking an arbitrary lift $(\mathbf{A}_i, \mathbf{b}_i) \in Q^{\text{dan}}$, then any other lift is of the form $(x_i \mathbf{A}_i, \mathbf{b}_i/x_i)$, for some $x_i \in \mathbb{R} \setminus 0$, $i = 1, 2$, and the horizontality condition is $(x_1 \mathbf{A}_1) \times (x_2 \mathbf{A}_2) = \mathbf{b}_2/x_2 - \mathbf{b}_1/x_1$.

Now $b_1 b_2 \in A_1 A_2$ implies $\mathbf{A}_1 \times \mathbf{A}_2 = \lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2$ for some $\lambda_i \neq 0$, therefore $x_1 x_2 (\lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2) = \mathbf{b}_2/x_2 - \mathbf{b}_1/x_1$. Thus, the horizontality condition becomes $(x_1)^2 x_2 \lambda_1 = -1$ and $x_1 (x_2)^2 \lambda_2 = 1$. This system has a unique solution $x_1 = \sqrt[3]{\lambda_2/(\lambda_1)^2}$, $x_2 = -\sqrt[3]{\lambda_1/(\lambda_2)^2}$, as needed. $\square$

Lemma 3.2 (Theorem 2.5 for $n = 3$). *Consider a horizontal non-degenerate 3-gon in Q^{dan}, given by 3 vertices $q_1, q_2, q_3 \in Q^{\text{dan}}$ such that $q_1 q_2$ and $q_2 q_3$ are horizontal lines. Then the projected pair of inscribed 3-gons with vertices A_1, A_2, A_3 and edges b_1, b_2, b_3 is a dancing pair; i.e., it satisfies*

$$[A_2, B_1, A_1, D] + [B_2, A_2, C, A_3] = 0, \quad (10)$$

where $B_1 = b_1 b_2, B_2 = b_2 b_3, C := b_1 a_2$ and $D := a_1 b_2$. Conversely, a dancing pair of 3-gons lifts uniquely to a horizontal 3-gon in Q^{dan}.
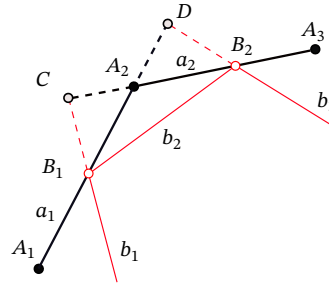
Proof. Let $q_i = (\mathbf{A}_i, \mathbf{b}_i), i = 1, 2, 3$. By Lemma 3.1, the projected pair of 3-gons is inscribed and we need to show that Equation (10) is satisfied. We give homogeneous coordinates to all points involved:

$$\mathbf{B}_1 := \mathbf{b}_1 \times \mathbf{b}_2, \ \mathbf{B}_2 := \mathbf{b}_2 \times \mathbf{b}_3, \ \mathbf{C} := \mathbf{b}_1 \times (\mathbf{A}_2 \times \mathbf{A}_3), \ \mathbf{D} := \mathbf{b}_3 \times (\mathbf{A}_1 \times \mathbf{A}_2).$$

We now write these expressions in terms of $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$. From the horizontality condition $\mathbf{b}_2 - \mathbf{b}_1 = \mathbf{A}_1 \times \mathbf{A}_2$ and the vector identity

$$\mathbf{a} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{a}\mathbf{C})\mathbf{B} - (\mathbf{a}\mathbf{B})\mathbf{C}, \quad (11)$$

follows

$$\mathbf{b}_1 \times \mathbf{b}_2 = \mathbf{b}_1 \times (\mathbf{b}_1 + \mathbf{A}_1 \times \mathbf{A}_2) = (\mathbf{b}_1 \mathbf{A}_2)\mathbf{A}_1 - \mathbf{A}_2.$$

Now $(\mathbf{A}_2 - \mathbf{A}_1, \mathbf{b}_2 - \mathbf{b}_1)$ is tangent to $Q^{\mathrm{dan}} = \{\mathbf{b}\mathbf{A} = 1\}$ at $(\mathbf{A}_1, \mathbf{b}_1)$, hence

$$0 = (\mathbf{b}_2 - \mathbf{b}_1)\mathbf{A}_1 + \mathbf{b}_1(\mathbf{A}_2 - \mathbf{A}_1) = \mathbf{b}_2\mathbf{A}_1 + \mathbf{b}_1\mathbf{A}_2 - 2 =$$
$$= (\mathbf{b}_1 - \mathbf{A}_1 \times \mathbf{A}_2)\mathbf{A}_1 + \mathbf{b}_1\mathbf{A}_2 - 2 = \mathbf{b}_1\mathbf{A}_2 - 1,$$

hence $\mathbf{B}_1 = \mathbf{A}_1 - \mathbf{A}_2$. Similarly, $\mathbf{B}_2 = \mathbf{A}_2 - \mathbf{A}_3$. Using again equation (11), $\mathbf{C} = (\mathbf{b}_1\mathbf{A}_3)\mathbf{A}_2 - \mathbf{A}_3$ and $\mathbf{D} = \mathbf{A}_1 - (\mathbf{b}_3\mathbf{A}_1)\mathbf{A}_2$.

Now it is easy to show that if 4 collinear points $A_1, \dots, A_4 \in \mathbb{RP}^2$ are given by homogeneous coordinates $\mathbf{A}_i \in \mathbb{R}^3$, such that $\mathbf{A}_3 = \mathbf{A}_1 + \mathbf{A}_2, \mathbf{A}_4 = k\mathbf{A}_1 + \mathbf{A}_2$, then $[A_1, A_2, A_3, A_4] = k$ (see for example [12, Theorem 74, page 105]). Using this formula and the above expressions for $\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}, \mathbf{D}$, we get

$$[A_2, B_1, A_1, D] = 1 - \mathbf{b}_3\mathbf{A}_1, \quad [B_2, A_2, C, A_3] = 1 - \mathbf{b}_1\mathbf{A}_3,$$

hence

$$[A_2, B_1, A_1, D] + [B_2, A_2, C, A_3] = 2 - \mathbf{b}_3\mathbf{A}_1 - \mathbf{b}_1\mathbf{A}_3.$$

Next,

$$\mathbf{b}_3\mathbf{A}_1 + \mathbf{b}_1\mathbf{A}_3 = (\mathbf{b}_2 + \mathbf{A}_2 \times \mathbf{A}_3)\mathbf{A}_1 + (\mathbf{b}_2 - \mathbf{A}_1 \times \mathbf{A}_2)\mathbf{A}_3$$
$$= \mathbf{b}_2\mathbf{A}_1 + \mathbf{b}_2\mathbf{A}_3 = 2,$$

so Equation (10) is satisfied, as needed.

In the other direction, suppose an inscribed pair of 3-gons is given, with vertices A_1, A_2, A_3 and edges b_1, b_2, b_3, satisfying Equation (10). By Lemma 3.1, we can uniquely lift $(A_1, b_1), (A_2, b_2)$ to points $(\mathbf{A}_1, \mathbf{b}_1), (\mathbf{A}_2, \mathbf{b}_2) \in Q^{\mathrm{dan}}$ on a horizontal line.

Likewise, we can uniquely lift $(A_2, b_2), (A_3, b_3)$ to points $(\lambda\mathbf{A}_2, \mathbf{b}_2/\lambda), (\mathbf{A}_3, \mathbf{b}_3)$ in Q^{dan} on a horizontal line, for some $\lambda \neq 0$. See Figure 9.
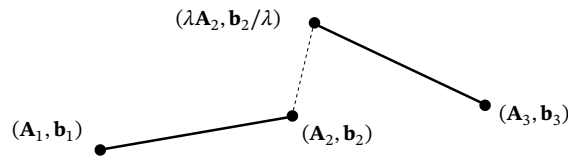

FIGURE 9. Lemma 3.2.

We now show that the dancing condition (10) implies $\lambda = 1$, i.e., the three lifted points $(\mathbf{A}_1, \mathbf{b}_1), (\mathbf{A}_2, \mathbf{b}_2), (\mathbf{A}_3, \mathbf{b}_3)$ are the vertices of a horizontal 3-gon.

We use the following:

- $\mathbf{b}_2 = \mathbf{b}_1 + \mathbf{A}_1 \times \mathbf{A}_2, \mathbf{b}_3 = \frac{1}{\lambda}\mathbf{b}_2 + \lambda\mathbf{A}_2 \times \mathbf{A}_3$;
- $A_1 = [\mathbf{A}_1], A_2 = [\mathbf{A}_2]$
- $B_1 = [\mathbf{b}_1 \times \mathbf{b}_2] = [\mathbf{A}_1 - \mathbf{A}_2], B_2 = [\mathbf{b}_2 \times \mathbf{b}_3] = [\lambda\mathbf{A}_2 - \mathbf{A}_3]$;
- $C = [\mathbf{b}_1 \times (\mathbf{A}_2 \times \mathbf{A}_3)] = [(\mathbf{b}_1\mathbf{A}_3)\mathbf{A}_2 - \mathbf{A}_3]$;
- $D = [\mathbf{b}_3 \times (\mathbf{A}_\times\mathbf{A}_2)] = [(\mathbf{b}_3\mathbf{A}_2)\mathbf{A}_1 - (\mathbf{b}_3\mathbf{A}_1)\mathbf{A}_2] = [\frac{1}{\lambda}\mathbf{A}_1 - (\mathbf{b}_3\mathbf{A}_1)\mathbf{A}_2]$.

A similar computation as above gives

$$[A_2, B_1, A_1, D] = 1 - \lambda \mathbf{b}_3 \mathbf{A}_1$$

$$[B_2, A_2, C, A_3] = 1 - \frac{1}{\lambda} \mathbf{b}_1 \mathbf{A}_3$$

Now, from $\mathbf{b}_3 = \frac{1}{\lambda} \mathbf{b}_2 + \lambda \mathbf{A}_2 \times \mathbf{A}_3$, we get that

$$\mathbf{b}_3 \mathbf{A}_1 = \frac{1}{\lambda} + \lambda (\mathbf{A}_1 \times \mathbf{A}_2) \mathbf{A}_3;$$

whereas, from $\mathbf{b}_2 = \mathbf{b}_1 + \mathbf{A}_1 \times \mathbf{A}_2$ we obtain

$$\mathbf{b}_1 \mathbf{A}_3 = \lambda - (\mathbf{A}_1 \times \mathbf{A}_2) \mathbf{A}_3,$$

since $\mathbf{b}_2 \mathbf{A}_3 = \lambda$.

Thus,

$$0 = [A_2, B_1, A_1, D] + [B_2, A_2, C, A_3]$$

$$= 2 - \lambda \mathbf{b}_3 \mathbf{A}_1 - \frac{1}{\lambda} \mathbf{b}_1 \mathbf{A}_3$$

$$= 2 - (1 + \lambda^2 (\mathbf{A}_1 \times \mathbf{A}_2) \mathbf{A}_3) - (1 - \frac{1}{\lambda} (\mathbf{A}_1 \times \mathbf{A}_2) \mathbf{A}_3)$$

$$= \left(\frac{1}{\lambda} - \lambda^2 \right) (\mathbf{A}_1 \times \mathbf{A}_2) \mathbf{A}_3$$

Since $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are non-collinear, $(\mathbf{A}_1 \times \mathbf{A}_2) \mathbf{A}_3 \neq 0$. Therefore, $\lambda = 1$. $\qquad\square$

Theorem 2.5 now follows from these previous two lemmas by induction on n, using Lemma 3.2 for the inductive step. $\qquad\square$

3.2. Theorem 1.9. We identify $\mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$, $(x, y, z) \mapsto x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and use repeatedly the following well known facts:

  (i) If $\mathbf{w} \in \mathbb{R}^3$ is a unit vector and $\theta \in \mathbb{R}$ then $\mathbf{v} \mapsto \mathbf{v}' = e^{\theta \mathbf{w}} \mathbf{v} e^{-\theta \mathbf{w}}$ is the rotation about the axis $\mathbb{R}\mathbf{w}$ by the angle 2θ, in the sense given by the 'right hand rule' ($\det(\mathbf{v}, \mathbf{v}', \mathbf{w}) > 0$).
  (ii) $e^{\theta \mathbf{w}} = \cos\theta + \mathbf{w}\sin\theta$.

Proof of Part (b). (Part (a) will follow easily from part (b).) Let $\mathbf{v}_0, ..., \mathbf{v}_{n-1}$ be the vertices of Γ, arranged on a circle of (spherical) radius ϕ, centered at the north pole $\mathbf{k}$ of the stationary sphere (which we assume here to be a unit sphere, to simplify notation). Let w be the winding number of Γ about $\mathbf{k}$, so that $\theta := 2\pi w/n$ is the angle of rotation at $\mathbf{k}$ sending $\mathbf{v}_i \mapsto \mathbf{v}_{i+1}$. That is, setting $q := e^{\theta \mathbf{k}/2}$, one has $\mathbf{v}_i = q^i \mathbf{v}_0 \bar{q}^i$. As we roll the moving sphere along the edge of Γ joining $\mathbf{v}_i$ to $\mathbf{v}_{i+1}$, an arc of a great circle of length δ, the moving sphere rotates about the axis through its center in the direction of

$$\mathbf{w}_i := \frac{\mathbf{v}_i \times \mathbf{v}_{i+1}}{\|\mathbf{v}_i \times \mathbf{v}_{i+1}\|}$$

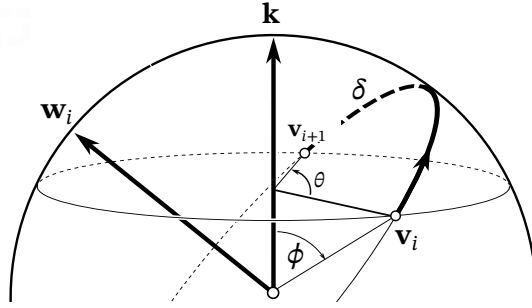by an angle of 4δ (due to the 3:1 radius ratio). See Fig. 10.

FIGURE 10.

Thus, the lifted monodromy due to rolling along this edge is $g_i := e^{2\delta \mathbf{w}_i}$, and the total lifted monodromy is

$$g := g_{n-1} \cdots g_1 g_0.$$

Now, clearly $\mathbf{w}_i = q^i \mathbf{w}_0 \bar{q}^i$, hence $g_i = q^i g_0 \bar{q}^i$, so

$$g = \left(q^{n-1} g_0 \bar{q}^{n-1}\right) \left(q^{n-2} g_0 \bar{q}^{n-2}\right) \cdots (q g_0 \bar{q}) \, g_0 = q^n (\bar{q} g_0)^n.$$

Next

$$q^n = e^{n\theta \mathbf{k}/2} = e^{n(2\pi w/n)\mathbf{k}/2} = e^{\pi \mathbf{k} w} = (-1)^w,$$

hence $g = (-1)^w (\bar{q} g_0)^n$. It follows that the trivial lifted monodromy condition, $g = 1$, is equivalent to

$$(\bar{q} g_0)^n = (-1)^w. \tag{12}$$

Lemma 3.3. *The condition $p^n = (-1)^w$, for a unit quaternion p, is equivalent to the existence of an integer $w' \equiv w \pmod 2$ such that $\mathrm{Re}(p) = \cos(\pi w'/n)$.*

Proof. Write $p = e^{t\mathbf{w}} = \cos t + \mathbf{w} \sin t$, for some unit imaginary quaternion $\mathbf{w}$ and $t \in \mathbb{R}$. Then $p^n = \cos(nt) + \mathbf{w} \sin(nt) = (-1)^w \Leftrightarrow \cos(nt) = (-1)^w \Leftrightarrow nt = \pi w'$ for some integer $w' \equiv w \pmod 2$. $\qquad \square$

To apply the last Lemma to $p = \bar{q} g_0$, we calculate its real part

$$\mathrm{Re}(\bar{q} g_0) = \mathrm{Re}\left(e^{-\theta \mathbf{k}/2} e^{2\delta \mathbf{w}_0}\right) =$$
$$= \cos(\theta/2) \cos(2\delta) + (\mathbf{k} \cdot \mathbf{w}_0) \sin(\theta/2) \sin(2\delta). \tag{13}$$

Next, we have

$$\mathbf{w}_0 = \frac{\mathbf{v}_0 \times \mathbf{v}_1}{\|\mathbf{v}_0 \times \mathbf{v}_1\|}, \quad \|\mathbf{v}_0 \times \mathbf{v}_1\| = \sin\delta, \quad (\mathbf{v}_0 \times \mathbf{v}_1) \cdot \mathbf{k} = \sin\theta \sin^2\phi,$$

thus

$$\mathbf{k} \cdot \mathbf{w}_0 = \frac{\sin^2\phi \sin\theta}{\sin\delta}.$$

As for δ, we consider the spherical right triangle with vertices $\mathbf{k}, \mathbf{v}_0, M$, where M is the midpoint of the edge of Γ joining $\mathbf{v}_0$ with $\mathbf{v}_1$. See Figure 11.

By standard spherical trigonometry (e.g., formula (R3) of [15]),

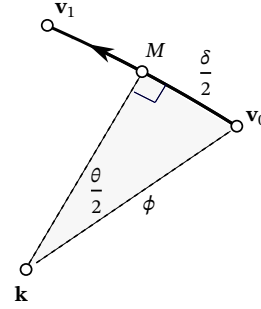$$\sin(\delta/2) = \sin\phi\sin(\theta/2).$$

FIGURE 11.

Using the last two displayed equations in (13), we obtain, after some simplification,

$$\mathrm{Re}(\bar{q}g_0) = \cos(\theta/2)\left[1 - 4\sin^2(\theta/2)\sin^2\phi\right].$$

Recalling that $\theta = 2\pi w/n$ and the lifted monodromy triviality condition $\mathrm{Re}(\bar{q}g_0) = \cos(\pi w'/n)$ for some $w' \equiv w \pmod 2$, we obtain Equation (3),

$$\cos(\pi w'/n) = \cos(\pi w/n)\left[1 - 4\sin^2(\pi w/n)\sin^2\phi\right]. \tag{14}$$

This completes the proof of part (b) of Theorem 1.9, except for the bound $w < w' < n$. This follows from the last equation: by the periodicity of the left hand side, one can assume, without loss of generality, that $0 \leq w' < n$. Now the right hand side is a strictly decreasing function of $\phi \in (0, \pi/2)$, taking values in the open interval $(\cos(3\pi w/n), \cos(\pi w/n))$. Thus, if $\cos(\pi w'/n)$ is one of these values, we must have that $\pi w/n < \pi w'/n$, or $w < w'$.

Proof of Part (a). This proof is very similar to part (b) above, except that for the trivial (unlifted) monodromy condition on Γ, one requires only that the lifted monodromy satisfies $g = \pm 1$, so in Lemma 3.3 one can drop the requirement $w' \equiv w \pmod 2$.

Proof of Part (c). Let Γ' be the regular polygon traced on the moving sphere as it is rolled along Γ. Its vertices are $\mathbf{v}'_0, \dots, \mathbf{v}'_{n-1}$, arranged on a circle of radius ϕ', with θ' the angle of rotation at the center, sending $\mathbf{v}'_i \mapsto \mathbf{v}'_{i+1}$. To show that $\mathbf{w}'$ in equation (3) is the winding number of Γ' about its center we thus need to show that $\theta' := 2\pi w'/n$.

We consider the shaded triangle Figure 11 and the corresponding triangle on the moving sphere. On the stationary sphere the angles are $\theta/2, \pi/2, A$, and the sides opposite the first two angles are $\delta/2, \phi$ (respectively). On the moving sphere the angles are $\theta'/2, \pi/2, A$, and the sides opposite the first two angles are $3\delta/2, \phi'$ (respectively). By formulas (R3) and (R9) of [15] applied to these

two triangles,

$$\cos(\theta/2) = \sin A \cos(\delta/2),$$
$$\cos(\theta'/2) = \sin A \cos(3\delta/2),$$
$$\sin(\delta/2) = \sin(\theta/2)\sin\phi.$$

Hence,

$$\cos(\theta'/2) = \cos(\theta/2)\frac{\cos(3\delta/2)}{\cos(\delta/2)} = \cos(\theta/2)\left[1 - 4\sin^2(\delta/2)\right]$$
$$= \cos(\theta/2)\left[1 - 4\sin^2(\theta/2)\sin^2\phi\right] = \cos\left(\pi w'/n\right).$$

Thus $\cos(\theta'/2) = \cos\left(\pi w'/n\right)$, hence $\theta' = 2\pi w'/n$, as needed.

Proof of Part (d). We first show that for all $n \geq 6$, $w = 2$ and $w' = 4$, Equation (19) has a solution, using an intermediate-value argument: let $\alpha = 2\pi/n$, then we need to show that $\cos(2\alpha) = \cos\alpha\left(1 - 4\sin^2\alpha\sin^2\phi\right)$ has a solution $\phi \in (0, \pi/2)$. As ϕ varies in $[0, \pi/2]$, the right hand side of this equation decreases monotonically from $\cos\alpha$ to $\cos\alpha\left(1 - 4\sin^2\alpha\right) = \cos(3\alpha)$. For $n \geq 6$, one has $3\alpha \leq \pi$, hence $\cos(\alpha) > \cos(2\alpha) > \cos(3\alpha)$, so there is an intermediate value of $\phi \in (0, \pi/2)$ for which the right hand side is $\cos(2\alpha)$.

To show that there is no solution of (19) with $n < 6$, we can either use Theorem 1.4, or more elementary, prove directly that there are no solutions to Equation (19) with $n < 6$, $0 < w < n/2$, $w < w' < n$ and $w' \equiv w \pmod 2$. There are only 3 cases of (n, w, w') satisfying these restrictions: $(4, 1, 3), (5, 1, 3)$ and $(5, 2, 4)$. In these 3 cases, it is easy to show that the equation reduces to $\cos\phi = 0$, so there is no solution $\phi \in (0, \pi/2)$. $\qquad\square$

3.3. Theorem 2.12. We divide the proof as follows:

(1) $\iota : Q^{\mathrm{dan}} \to Q^{\mathrm{oct}}$ is an embedding, mapping $\mathscr{D}^{\mathrm{dan}}$ to $\mathscr{D}^{\mathrm{oct}}$.
(2) $\Phi : \widetilde{Q}^{\mathrm{roll}} \to \widetilde{Q}^{\mathrm{oct}}$ is a diffeomorphism
(3) Φ maps $\widetilde{\mathscr{D}}^{\mathrm{roll}}$ to $\widetilde{\mathscr{D}}^{\mathrm{oct}}$.

Proof of (1). This was shown in [4, §3.3], with slightly different notation, so we sketch the proof here. The map $\mathbb{R}^{3,3} \to \mathbb{P}(\mathrm{Im}(\mathbb{O}))$, $(\mathbf{A}, \mathbf{b}) \mapsto [\zeta]$, where $\zeta = \left(\begin{smallmatrix} 1 & \mathbf{A} \\ \mathbf{b} & -1 \end{smallmatrix}\right)$, is clearly injective (an affine chart). If $(\mathbf{A}, \mathbf{b}) \in Q^{\mathrm{dan}}$, i.e., $\mathbf{bA} = 1$, then $\langle \zeta, \zeta \rangle = -1 + \mathbf{bA} = 0$, i.e. $\iota(Q^{\mathrm{dan}}) \subset Q^{\mathrm{oct}}$.

Next, let $\Omega := \zeta\,\mathrm{d}\zeta$, an $\mathbb{O}$-valued 1-form on $\mathrm{Im}(\mathbb{O})$. Explicitly,

$$\Omega = \begin{pmatrix} x\,\mathrm{d}x - \mathrm{d}\mathbf{b}\,\mathbf{A} & x\,\mathrm{d}\mathbf{A} - \mathbf{A}\,\mathrm{d}x + \mathbf{b}\times\mathrm{d}\mathbf{b} \\ \mathbf{b}\,\mathrm{d}x - x\mathrm{d}\mathbf{b} + \mathbf{A}\times\mathrm{d}\mathbf{A} & x\,\mathrm{d}x - \mathbf{b}\,\mathrm{d}\mathbf{A} \end{pmatrix}. \tag{15}$$

Next, one calculates that $r^*\Omega = r^2\Omega$, $r \in \mathbb{R}^*$, and that Ω vanishes along the radial directions in the null cone $C \subset \mathrm{Im}(\mathbb{O})$, hence the restriction of Ω to C descends to the quotient $(C \setminus 0)/\mathbb{R}^* = Q^{\mathrm{oct}}$, with kernel $\mathscr{D}^{\mathrm{oct}}$ (see [4, Proposition 3.5] for the detailed calculation). It is thus enough to show that the

kernel of the pull-back of Ω to Q^{dan} by $\tilde{\iota} : (\mathbf{A}, \mathbf{b}) \mapsto \left(\begin{smallmatrix} 1 & \mathbf{A} \\ \mathbf{b} & -1 \end{smallmatrix} \right)$ is $\mathscr{D}^{\mathrm{dan}}$. Indeed, from Equation (15) follows

$$\tilde{\iota}^* \Omega = \begin{pmatrix} -\mathrm{d}\mathbf{b}\,\mathbf{A} & \mathrm{d}\mathbf{A} + \mathbf{b} \times \mathrm{d}\mathbf{b} \\ -\mathrm{d}\mathbf{b} + \mathbf{A} \times \mathrm{d}\mathbf{A} & -\mathbf{b}\,\mathrm{d}\mathbf{A} \end{pmatrix},$$

then one checks that the common kernel of the entries of this 1-form, restricted to Q^{dan}, is indeed $\mathscr{D}^{\mathrm{dan}}$. $\qquad\square$

Proof of (2). We express $\Phi : \widetilde{Q}^{\mathrm{roll}} \to \widetilde{Q}^{\mathrm{oct}}$ as the composition

$$S^2 \times S^3 \xrightarrow{f} S^2 \times S^3 \xrightarrow{j} C \setminus 0 \xrightarrow{\pi} \widetilde{Q}^{\mathrm{oct}}, \tag{16}$$

where

- f is the restriction to $S^2 \times S^3$ of the map
$$\mathrm{Im}(\mathbb{H}) \times \mathbb{H} \to \mathrm{Im}(\mathbb{H}) \times \mathbb{H}, \ \ (\mathbf{v}, q) \mapsto (\mathbf{v}, \mathbf{v}q),$$

- j is the restriction to $S^2 \times S^3$ of the linear isomorphism
$$\mathrm{Im}(\mathbb{H}) \times \mathbb{H} \to \mathrm{Im}(\mathbb{O}), \ \ (\mathbf{v}, x + \mathbf{w}) \mapsto \begin{pmatrix} x & \mathbf{v} + \mathbf{w} \\ \mathbf{v} - \mathbf{w} & -x \end{pmatrix}, \tag{17}$$

  where $\mathbf{w}, \mathbf{v} \in \mathrm{Im}(\mathbb{H})$, $x \in \mathbb{R}$,
- $C = \{\zeta \in \mathrm{Im}(\mathbb{O}) \mid \langle \zeta, \zeta \rangle = 0\}$ is the *null cone* in $\mathrm{Im}(\mathbb{O})$, and
- π is the restriction to $C \setminus 0$ of the canonical projection
$$\mathrm{Im}(\mathbb{O}) \setminus 0 \to (\mathrm{Im}(\mathbb{O}) \setminus 0)/\mathbb{R}^+ = S^6, \ \ \zeta \mapsto \mathbb{R}^+ \zeta.$$

Remark 3.4. We note, as we did after Equation (9), that for Equation (17) to make sense we need to identify $\mathbb{R}^3$ (column vectors) with $(\mathbb{R}^3)^*$ (row vectors) using the standard Euclidean inner product in $\mathbb{R}^3$. This convention will be kept implicitly for the rest of the article.

First, f is a diffeomorphism because it has an inverse,

$$(\mathbf{v}, q) \mapsto (\mathbf{v}, -\mathbf{v}q).$$

Next, we verify that j maps $S^2 \times S^3$ into $C \setminus 0$: if $(\mathbf{v}, x + \mathbf{w}) \in S^2 \times S^3$, then $|\mathbf{v}|^2 = x^2 + |\mathbf{w}|^2 = 1$, hence if $\zeta = j(\mathbf{v}, x + \mathbf{w}) = \left(\begin{smallmatrix} x & \mathbf{v}+\mathbf{w} \\ \mathbf{v}-\mathbf{w} & -x \end{smallmatrix} \right)$ then $\langle \zeta, \zeta \rangle = -x^2 + (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = |\mathbf{v}|^2 - (x^2 + |\mathbf{w}|^2) = 1 - 1 = 0$, hence $j(\mathbf{v}, x + \mathbf{w}) \in C$. Now 0 is clearly not in the image, since j is the restriction of an injectve linear map and 0 is not in $S^2 \times S^3$.

Next, to show that $\pi \circ j$ is a diffeomorphism, we show that it is bijective and a local diffeomorphism.

To show that $\pi \circ j$ is injective, note that j is injective and its image intersects each of the fibers of π at a single point.

To show that $\pi \circ j$ is surjective, let $[\zeta] \in \widetilde{Q}^{\mathrm{oct}}$. Then $\zeta = \left(\begin{smallmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{smallmatrix} \right) \in C \setminus 0$, hence $-x^2 + \mathbf{b}\mathbf{A} = 0$. Let $\mathbf{v} := (\mathbf{A} + \mathbf{b})/2$, $\mathbf{w} := (\mathbf{A} - \mathbf{b})/2$, then $|x + \mathbf{w}|^2 = x^2 + |\mathbf{w}|^2 = x^2 + |\mathbf{A} - \mathbf{b}|^2/4 = x^2 + |\mathbf{A} + \mathbf{b}|^2/4 - \mathbf{b}\mathbf{A} = |\mathbf{v}|^2$. Thus, $|x + \mathbf{w}| = |\mathbf{v}| \neq 0$, so we can (positively) rescale $\mathbf{v}$ and $x + \mathbf{w}$ simultaneously to unit vectors.

To show that $\pi \circ j$ is a local diffeomorphism, it is enough to show, by the inverse function theorem, that its differential $(\pi \circ j)_*$ is bijective at each point of $S^2 \times S^3$.

Now j is the restriction of a linear isomorphism, hence is an immersion, i.e., j_* is injective at each point of $S^2 \times S^3$, and π is a submersion, i.e., π_* is surjective at each point of $C \setminus 0$, with kernel in the radial direction, transverse to the image of j_*. Hence, the composition $\pi_* \circ j_* = (\pi \circ j)_*$ is bijective, as needed. □

Proof of (3). We shall prove this statement in several steps:

(a) Define a transitive action of $K := S^3 \times S^3$ on $\widetilde{Q}^{\mathrm{roll}}$ and $\widetilde{Q}^{\mathrm{oct}}$.
(b) $\Phi : \widetilde{Q}^{\mathrm{roll}} \to \widetilde{Q}^{\mathrm{oct}}$ is K-equivariant.
(c) $\widetilde{\mathscr{D}}^{\mathrm{roll}}, \widetilde{\mathscr{D}}^{\mathrm{oct}}$ are K-invariant.
(d) Φ_* maps $\widetilde{\mathscr{D}}^{\mathrm{roll}}$ to $\widetilde{\mathscr{D}}^{\mathrm{oct}}$ at a single point of $\widetilde{Q}^{\mathrm{roll}}$.

We proceed with proofs of each of these steps.

Step (a). Define a linear action of $K = S^3 \times S^3$ (pairs of unit quaternions) on $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}$ via

$$(q_1, q_2) : (\mathbf{v}, q) \mapsto (q_1 \mathbf{v} \bar{q}_1, q_1 q \bar{q}_2). \tag{18}$$

This leaves invariant $\widetilde{Q}^{\mathrm{roll}} = S^2 \times S^3$ so defines a K-action on it, clearly transitive.

This linear K-action on $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}$ also induces a K-action on $\mathrm{Im}(\mathbb{O})$ via the linear isomorphism $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H} \to \mathrm{Im}(\mathbb{O})$ of formula (17). Under this isomorphism, $\langle \, , \, \rangle$ becomes the quadratic form $|\mathbf{v}|^2 - |q|^2$ on $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}$, which is clearly K-invariant, hence the null cone $C \subset \mathrm{Im}(\mathbb{O})$ is K-invariant, inducing a K-action on $\widetilde{Q}^{\mathrm{oct}} = (C \setminus 0)/\mathbb{R}^+$.

Step (b). Each of the maps f, ι, π of (16) are K-equivariant, hence so is Φ. The (easy) verification is left to the reader.

Step (c). We will use throughout a well-known (and easy to verify) quaternion identity:

$$\mathbf{v}\mathbf{w} = -\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \times \mathbf{w}, \quad \forall \mathbf{v}, \mathbf{w} \in \mathrm{Im}(\mathbb{H}).$$

We start with the K-invariance of $\widetilde{\mathscr{D}}^{\mathrm{roll}}$. Note that the K-action on $\widetilde{Q}^{\mathrm{roll}}$ commutes with that of $(\pm 1, \pm 1)$, hence it descends to an action of $K/(\pm 1, \pm 1) = \mathrm{SO}_3 \times \mathrm{SO}_3$ on the quotient $Q^{\mathrm{roll}} = \widetilde{Q}^{\mathrm{roll}}/(1, \pm 1) = S^2 \times \mathrm{SO}_3, (g_1, g_2) : (\mathbf{v}, g) \mapsto (g_1 \mathbf{v}, g_1 g g_2^{-1})$. Thus, in order to show that $\widetilde{\mathscr{D}}^{\mathrm{roll}}$ is K-invariant it is enough to show that $\mathscr{D}^{\mathrm{roll}}$ is $\mathrm{SO}_3 \times \mathrm{SO}_3$-invariant. Now at a point $(\mathbf{v}, g) \in Q^{\mathrm{roll}} = S^2 \times \mathrm{SO}_3, \mathscr{D}^{\mathrm{roll}}$ consists of vectors $(\dot{\mathbf{v}}, \dot{g}) \in T_{(\mathbf{v}, g)} Q^{\mathrm{roll}}$ satisfying

$$4\dot{\mathbf{v}} = \omega \times \mathbf{v}, \quad \omega \cdot \mathbf{v} = 0, \tag{19}$$

where $\omega \times \mathbf{x} = \dot{g} g^{-1} \mathbf{x}$, see Definition 2.6. Now $(g_1, g_2)_* : (\dot{\mathbf{v}}, \dot{g}) \mapsto (g_1 \dot{\mathbf{v}}, g_1 \dot{g} g_2^{-1})$, thus

$$g_1 \dot{g} g_2^{-1} (g_1 g g_2^{-1})^{-1} \mathbf{x} = g_1 \dot{g} g^{-1} g_1^{-1} \mathbf{x} = g_1 (\omega \times g_1^{-1} \mathbf{x}) = (g_1 \omega) \times \mathbf{x},$$

hence $(g_1, g_2)_* : (\dot{\mathbf{v}}, \boldsymbol{\omega}) \mapsto (g_1 \dot{\mathbf{v}}, g_1 \boldsymbol{\omega})$. Now if $(\dot{\mathbf{v}}, \boldsymbol{\omega})$ satisfy Equations (19) then $4g_1\dot{\mathbf{v}} = 4g_1(\boldsymbol{\omega} \times \mathbf{v}) = 4(g_1\boldsymbol{\omega}) \times (g_1\mathbf{v})$, $(g_1\boldsymbol{\omega}) \cdot (g_1\mathbf{v}) = \boldsymbol{\omega} \cdot \mathbf{v} = 0$, hence $(g_1\dot{\mathbf{v}}, g_1\boldsymbol{\omega})$ satisfy them as well. This shows that $\widetilde{\mathscr{D}}^{\mathrm{roll}}$ is K-invariant.

To show that $\widetilde{\mathscr{D}}^{\mathrm{oct}}$ is K-invariant, it is enough to show that the K-action defined on $\mathrm{Im}(\mathbb{O})$ by the isomorphism $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H} \to \mathrm{Im}(\mathbb{O})$ of formula (17) preserves octonion multiplication. For this, we show that the image of the infinitesimal K-action on $\mathrm{Im}(\mathbb{O})$ is contained in the Lie algebra $\mathfrak{g}_2 \subset \mathrm{End}(\mathrm{Im}(\mathbb{O}))$ defined by Equation (25).

To this end, let $(\mathbf{v}_1, \mathbf{v}_2) \in \mathrm{Im}(\mathbb{H}) \oplus \mathrm{Im}(\mathbb{H})$, thought of as the Lie algebra of $K = S^3 \times S^3$. The infinitesimal action of $(\mathbf{v}_1, \mathbf{v}_2)$ on $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}$, corresponding to the K-action of Equation (18), is

$$(\mathbf{v}, q) \mapsto (2\mathbf{v}_1 \times \mathbf{v}, \mathbf{v}_1 q - q\mathbf{v}_2).$$

Conjugating this action with the ismorphism $\mathrm{Im}(\mathbb{H}) \oplus \mathrm{Im}(\mathbb{H}) \to \mathrm{Im}(\mathbb{O})$ of Equation (17), we obtain the infinitesimal action of $(\mathbf{v}_1, \mathbf{v}_2)$ on $\mathrm{Im}(\mathbb{O})$,

$$\begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} & \tilde{\mathbf{A}} \\ \tilde{\mathbf{b}} & -\tilde{x} \end{pmatrix},$$

where

$$\begin{cases} \tilde{\mathbf{A}} &= \frac{1}{2}(3\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{A} + \frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2) \times \mathbf{b} + (\mathbf{v}_1 - \mathbf{v}_2)x \\[2mm] \tilde{\mathbf{b}} &= \frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2) \times \mathbf{A} + \frac{1}{2}(3\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{b} + (\mathbf{v}_2 - \mathbf{v}_1)x \\[2mm] \tilde{x} &= \frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{A} - \mathbf{b}) \end{cases}$$

(to simplify notation, all vectors in this formula are column vectors). One can see easily that this is the action of $\rho(T, \mathbf{Q}, \mathbf{p})/2$ of Equation (25), where

$$T\mathbf{x} = (3\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{x}, \ \mathbf{Q} = \mathbf{v}_1 - \mathbf{v}_2, \ \mathbf{p} = \mathbf{v}_2 - \mathbf{v}_1. \tag{20}$$

This concludes the proof of Step (c). $\qquad \square$

Step (d). We first find equations for $\widetilde{\mathscr{D}}^{\mathrm{roll}}$ on $\widetilde{Q}^{\mathrm{roll}} = S^2 \times S^3 \subset \mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}$, using coordinates $\mathbf{v} \in \mathrm{Im}(\mathbb{H})$, $s + \mathbf{w} \in \mathbb{H}$.

Lemma 3.5. $\widetilde{\mathscr{D}}^{\mathrm{roll}} \subset T\widetilde{Q}^{\mathrm{roll}}$ *is given by*

$$2d\mathbf{v} + \mathbf{v} \times (s\,d\mathbf{w} - \mathbf{w}\,ds + \mathbf{w} \times d\mathbf{w}) = 0 \qquad \text{(no slip)},$$
$$\mathbf{v} \cdot (s\,d\mathbf{w} - \mathbf{w}\,ds + \mathbf{w} \times d\mathbf{w}) = 0 \qquad \text{(no twist)}.$$

Proof. Recall first the equations that define $\mathscr{D}^{\mathrm{roll}}$ (Definition 2.6, for $\rho = 3$):

$$4\dot{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{v} \ \text{(no slip)}, \quad \boldsymbol{\omega} \cdot \mathbf{v} = 0 \ \text{(no twist)},$$

where $\boldsymbol{\omega}$ is defined by $\dot{g}g^{-1}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$.

Now let $q = q(t) \in S^3$, then $1 = q\bar{q}$ implies $0 = \dot{q}\bar{q} + q\dot{\bar{q}}$, i.e., $\dot{q}\bar{q} \in \mathrm{Im}(\mathbb{H})$. Next, define $g = \mathrm{Ad}(q)$, i.e., $\mathbf{x} = g\mathbf{X} = q\mathbf{X}\bar{q}$, where $\mathbf{X} \in \mathbb{R}^3$ (fixed). Then, on the one hand,

$$\dot{\mathbf{x}} = \dot{g}\mathbf{X} = \dot{g}g^{-1}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x},$$

and on the other hand,

$$\dot{\mathbf{x}} = \dot{q}\mathbf{X}\bar{q} + q\mathbf{X}\dot{\bar{q}} = \dot{q}\bar{q}\mathbf{x}q\bar{q} + q\bar{q}\mathbf{x}q\dot{\bar{q}} = \dot{q}\bar{q}\mathbf{x} - \mathbf{x}\dot{q}\bar{q} = 2\dot{q}\bar{q} \times \mathbf{x},$$

hence $\boldsymbol{\omega} = 2\dot{q}\bar{q}$.

Next, writing $q = s + \mathbf{w}$, one has $1 = q\bar{q} = s^2 + |\mathbf{w}|^2$, hence $0 = s\dot{s} + \mathbf{w} \cdot \dot{\mathbf{w}}$, thus

$$\begin{aligned}
\dot{q}\bar{q} &= (\dot{s} + \dot{\mathbf{w}})(s - \mathbf{w}) = \dot{s}s - \dot{s}\mathbf{w} + \dot{\mathbf{w}}s - \dot{\mathbf{w}}\mathbf{w} \\
&= -\dot{\mathbf{w}} \cdot \mathbf{w} - \dot{s}\mathbf{w} + \dot{\mathbf{w}}s - (-\dot{\mathbf{w}} \cdot \mathbf{w} + \dot{\mathbf{w}} \times \mathbf{w}) \qquad (21) \\
&= s\dot{\mathbf{w}} - \mathbf{w}\dot{s} + \mathbf{w} \times \dot{\mathbf{w}}.
\end{aligned}$$

It follows that the no slip equation on Q^{roll}, $4\dot{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{v}$, pulled back to $\widetilde{Q}^{\mathrm{roll}}$ by $q \mapsto \mathrm{Ad}(q)$, is

$$2\dot{\mathbf{v}} + \mathbf{v} \times \dot{q}\bar{q} = 2\dot{\mathbf{v}} + \mathbf{v} \times (s\dot{\mathbf{w}} - \mathbf{w}\dot{s} + \mathbf{w} \times \dot{\mathbf{w}}) = 0.$$

The no-twist equation also follows from the above expression (21) for $\dot{q}\bar{q}$,

$$\dot{q}\bar{q} \cdot \mathbf{v} = \mathbf{v} \cdot (s\dot{\mathbf{w}} - \mathbf{w}\dot{s} + \mathbf{w} \times \dot{\mathbf{w}}) = 0,$$

as needed.   □

Proof of step (d) continued. Now fix the point $(\mathbf{i}, 1) \in \widetilde{Q}^{\mathrm{roll}}$. From the last lemma follows that $\widetilde{\mathscr{D}}^{\mathrm{roll}}$ at this point is given by

$$dw_1 = dv_1 = 2dv_2 - dw_3 = 2dv_3 + dw_2 = 0. \qquad (22)$$

Next, consider $\Phi(\mathbf{i}, 1) = [\begin{pmatrix} 0 & 2\mathbf{i} \\ 0 & 0 \end{pmatrix}] \in \widetilde{Q}^{\mathrm{oct}}$. According to Equation (15), at this point $\widetilde{\mathscr{D}}^{\mathrm{oct}}$, pulled-back to the tangent to $C \subset \mathrm{Im}(\mathbb{H})$ at $\begin{pmatrix} 0 & 2\mathbf{i} \\ 0 & 0 \end{pmatrix}$, is given by

$$dx = db_1 = dA_2 = dA_3 = 0. \qquad (23)$$

Now $\Phi = \pi \circ \iota \circ f$, and $\iota \circ f$ is given by $(\mathbf{v}, s + \mathbf{w}) \mapsto \begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{pmatrix}$, where

$$\mathbf{A} = (1+s)\mathbf{v} + \mathbf{v} \times \mathbf{w}, \ \mathbf{b} = (1-s)\mathbf{v} - \mathbf{v} \times \mathbf{w}, \ x = -\mathbf{v} \cdot \mathbf{w}.$$

The pull back of Equations (23) under this map are

$$dw_1 = ds = 2dv_2 - dw_3 = 2dv_3 + dw_2 = 0.$$

This coincides with Equations (22), modulo the tangency equations to $S^2 \times S^3$ at $(\mathbf{i}, 1)$, $dv_1 = ds = 0$.   □

3.4. Theorem 1.13. The proof is based on Theorem 2.12. We need to establish first two lemmas.

To state the first lemma, we consider an ordered pair of points $([\mathbf{v}_1], [\mathbf{v}_2]) \in (S^2/\pm 1) \times (S^2/\pm 1)$, and associate with it a rolling monodromy $\mu \in S^3$, as follows. Consider a simple geodesic segment from $[\mathbf{v}_1]$ to $[\mathbf{v}_2]$ (there are 2 such segments), lift it to a spherical segment in S^2 (there are 2 such lifts), roll the moving sphere along this segment, then take the resulting lifted rolling monodromy $\mu \in S^3$.

Lemma 3.6. *The lifted rolling monodromy $\mu \in S^3$ depends only on the ordered pair $([\mathbf{v}_1], [\mathbf{v}_2]) \in (S^2 / \pm 1) \times (S^2 / \pm 1)$ and not on the various choices made.*

Proof. Let $\delta := \mathrm{dist}([\mathbf{v}_1], [\mathbf{v}_2])$, $0 < \delta \leq \pi/2$. There are two directed geodesic segments in $S^2 / \pm 1$ connecting $[\mathbf{v}_1]$ to $[\mathbf{v}_2]$, of lengths $\delta, \pi - \delta$. Each has two possible lifts to geodesic segments in S^2, of the same length, a total of 4 possibilities, with endpoints (i) $\mathbf{v}_1, \mathbf{v}_2$, (ii) $\mathbf{v}_1, -\mathbf{v}_2$, (iii) $-\mathbf{v}_1, -\mathbf{v}_2$, (iv) $-\mathbf{v}_1, \mathbf{v}_2$, of lengths $\delta, \pi - \delta, \delta, \pi - \delta$ (respectively). See Figure 12.
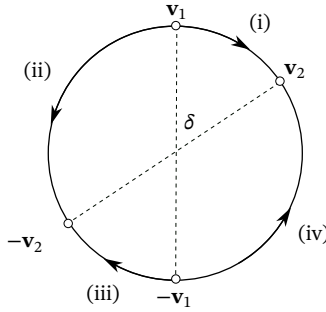

FIGURE 12.

Let us take case (i). Let $\mathbf{w} = \mathbf{v}_1 \times \mathbf{v}_2 / \|\mathbf{v}_1 \times \mathbf{v}_2\|$. As we roll the small sphere along the spherical arc segment of length δ from $\mathbf{v}_1$ to $\mathbf{v}_2$, it rotates about the $\mathbf{w}$ axis by an angle of 4δ. Thus, $\mu = e^{2\mathbf{w}\delta}$. In case (ii), the arc length changes to $\pi - \delta$ and $\mathbf{w}$ changes to $-\mathbf{w}$. Thus, $\mu = e^{2(\pi - \delta)(-\mathbf{w})} = e^{2\mathbf{w}\delta}$, same as in case (i). Cases (iii) and (iv) are analyzed similarly. $\qquad\square$

The second lemma concerns the map $\Phi : \widetilde{Q}^{\mathrm{roll}} \to \widetilde{Q}^{\mathrm{oct}}$ of Equation (9).

Lemma 3.7. *Let $(\mathbf{v}, q) \in \widetilde{Q}^{\mathrm{roll}}$, $[\zeta] = \Phi(\mathbf{v}, q) \in \widetilde{Q}^{\mathrm{oct}}$. Then $\Phi(-\mathbf{v}, q) = [-\zeta]$.*

In other words, Φ is $\mathbb{Z}_2$-equivariant, with respect to the actions $(-1) \times 1, -1$ on $\widetilde{Q}^{\mathrm{roll}}, \widetilde{Q}^{\mathrm{oct}}$ (respectively). The proof is immediate from formula (9). $\qquad\square$

Now we proceed to the bijective correspondence indicated in the statement of Theorem 1.13. We start with an equivalence class $[\Gamma]$ of non-degenerate closed spherical n-gons with trivial lifted rolling monodromy and an element $q \in S^3$. We associate with $([\Gamma], q)$ a non-degenerate closed horizontal n-gon in Q^{oct}, as follows. $[\Gamma]$ is given by the projected vertices $[\mathbf{v}_1], \dots, [\mathbf{v}_n] \in S^2 / \pm 1$ (see the first paragraph after Definition 1.12). We pick simple edges between succesive vertices (there are two possibilities for each successive pair) and obtain a closed polygonal path in $S^2 / \pm 1$. This path is then lifted to a spherical n-gon Γ in S^2 (there are two such lifts), with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots$, closed up to sign, i.e., $\mathbf{v}_{i+n} = \pm \mathbf{v}_i$ (same sign for all i). Let $\mu_i \in S^3$ be the lifted rolling monodromy along the edge of Γ from $\mathbf{v}_i$ to $\mathbf{v}_{i+1}$. By Lemma 3.6, μ_i depends only on $[\mathbf{v}_i], [\mathbf{v}_{i+1}]$.

Next, we lift Γ horizontally to $\widetilde{Q}^{\mathrm{roll}}$, starting at $(\mathbf{v}_1, q)$, and obtain a horizontal polygon, with vertices $(\mathbf{v}_1, q_1), \dots (\mathbf{v}_n, q_n)$, where $q_1 := q, q_2 := \mu_1 q_1, \dots, q_n =$

$\mu_{n-1} q_{n-1}$. This polygon is closed up to sign of $\mathbf{v}_i$, i.e., $(\mathbf{v}_{i+n}, q_{i+n}) = (\pm\mathbf{v}_i, q_i)$ (same sign for all i, depending on the choices made along the way).

This horizontal polygon is mapped by Φ to a horizontal polygon in $\widetilde{Q}^{\mathrm{oct}}$ with vertices $[\zeta_i] := \Phi[(\mathbf{v}_i, q_i)]$, $i = 1, \dots, n$. By Lemma 3.7, this polygon is closed up to sign, i.e., $[\zeta_{i+n}] = [\pm\zeta_i]$ (same sign for all i). The projection of this polygon to Q^{oct} is thus closed and horizontal, and its vertices do not depend on the choices made along the way (the edges do depend on the initial choice of edges in $S^2/\pm 1$, but a dancing pair of polygons is specified by its vertices alone). See Figure 13.
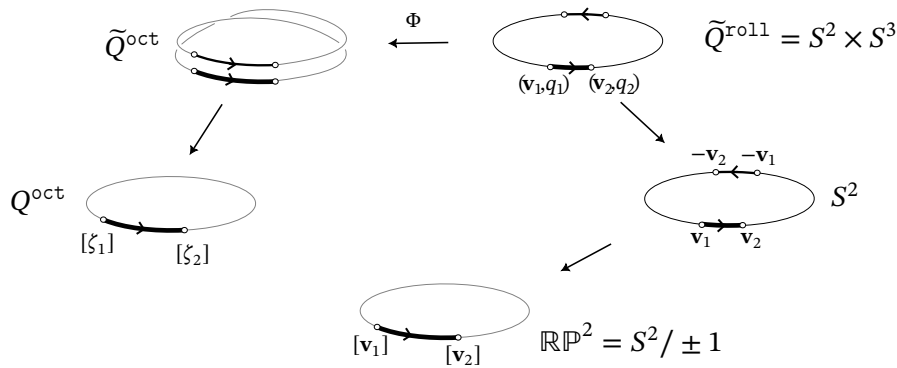

FIGURE 13.

If we add the genericity condition to $([\Gamma], q)$, then the n vertices in Q^{oct} lie in $Q^{\mathrm{oct}}_* = \iota(Q^{\mathrm{dan}})$, i.e., correspond to a dancing pair of polygons as per Theorem 2.5.

This correspondence is clearly invertible. Starting with a dancing pair of closed polygons, we first lift it to a horizontal polygon in Q^{dan} (by Theorem 2.5), then map it to a closed horizontal polygon in Q^{oct}_* (by step (1) of the proof of Theorem 2.12, see Section 3.3), lift to a horizontal polygon in $\widetilde{Q}^{\mathrm{oct}}_*$, then map by Φ^{-1} to a horizontal polygon in $\widetilde{Q}^{\mathrm{roll}}_*$, with vertices $(\mathbf{v}_1, q_1), \dots, (\mathbf{v}_n, q_n)$, closed up to sign of $\mathbf{v}_i$. It follows that the spherical polygon Γ with vertices $\mathbf{v}_1, \dots, \mathbf{v}_n$ has trivial lifted rolling monodromy and that $([\Gamma], q_1)$ is generic. $\qquad\square$

3.5. Theorem 1.4. From the correspondence of Theorem 2.5 (proved in Section 3.1), it is enough to show that (1) there exist non-degenerate horizontal n-gons in Q^{dan} for every $n \geq 6$, and (2) every horizontal n-gon in Q^{dan} for $n \leq 5$ is degenerate. The first statement follows from Corollary 1.10 and Theorem 1.13. We procced to prove the second statement.

Let $q_1, q_2, q_3 \in Q^{\mathrm{dan}}$ be the vertices of an open non-degenerate horizontal 3-gon; i.e., the three points are distinct and the lines $q_1 q_2$ and $q_2 q_3$ are horizontal and distinct. We shall prove that

(1) $q_1 q_3$ is not horizontal; therefore, there are no non-degenerate horizontal triangles.

(2) If $q_4 \in Q^{\text{dan}}$ is such that $q_1 q_4$ and $q_3 q_4$ are horizontal, then $q_4 = q_2$; therefore, there are no non-degenerate horizontal quadrilaterals.

(3) If $q_4, q_5 \in Q^{\text{dan}}$ are such that $q_3 q_4$, $q_4 q_5$ and $q_5 q_1$ are horizontal, then either $q_4 = q_2$ or $q_5 = q_2$; in either case, the pentagon will be a degenerate one.

We proceed with proofs of (1) and (3). The proof of (2) is similar to (3), slightly simpler, and is omitted.

Proof of (1). Suppose q_1, q_2, q_3 are 3 distinct points in Q^{dan} forming a horizontal triangle. We show that it is degenerate, i.e., the 3 points are collinear. Let $q_i = (\mathbf{A}_i, \mathbf{b}_i)$, $i = 1, 2, 3$. Then, by Definition 2.1,

$$\mathbf{b}_i - \mathbf{b}_j = \mathbf{A}_j \times \mathbf{A}_i, \ i, j = 1, 2, 3. \tag{24}$$

We can assume that $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are distinct: if $\mathbf{A}_i = \mathbf{A}_j$, $i \neq j$, then, by Equation (24), $\mathbf{b}_i - \mathbf{b}_j = \mathbf{A}_j \times \mathbf{A}_i = 0$, hence $q_i = q_j$, a contradiction.

Next, again by (24),

$$(\mathbf{A}_1 - \mathbf{A}_3) \times (\mathbf{A}_2 - \mathbf{A}_3) = \mathbf{A}_1 \times \mathbf{A}_2 - \mathbf{A}_1 \times \mathbf{A}_3 - \mathbf{A}_3 \times \mathbf{A}_2 =$$
$$= (\mathbf{b}_2 - \mathbf{b}_1) - (\mathbf{b}_3 - \mathbf{b}_1) - (\mathbf{b}_2 - \mathbf{b}_3) = 0,$$

so $\mathbf{A}_1 - \mathbf{A}_3 = \lambda(\mathbf{A}_2 - \mathbf{A}_3)$ for some $\lambda \neq 0$. Taking the cross product with $\mathbf{A}_3$, $\mathbf{A}_1 \times \mathbf{A}_3 = \lambda(\mathbf{A}_2 \times \mathbf{A}_3)$, which implies, by (24), $\mathbf{b}_1 - \mathbf{b}_3 = \lambda(\mathbf{b}_2 - \mathbf{b}_3)$, so $q_1 - q_3 = \lambda(q_2 - q_3)$, hence q_1, q_2, q_3 are collinear.

Proof of (3). Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis of $\mathbb{R}^3$ and $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ the dual basis. Since $\text{SL}_3(\mathbb{R})$ acts transitively on Q^{dan} preserving horizontality, we may assume, without loss of generality, that $q_2 = (\mathbf{e}_1, \mathbf{e}^1)$. The isotropy group at this point is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in \text{SL}_2(\mathbb{R}) \right\},$$

acting on the affine plane $\mathscr{D}_{q_2}^{\text{dan}} = \text{Span}\{(\mathbf{e}_2, \mathbf{e}^3), (\mathbf{e}_3, -\mathbf{e}^2)\}$. Hence, without loss of generality, we may assume that

$$q_1 = q_2 + (\mathbf{e}_2, \mathbf{e}^3) = (\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}^1 + \mathbf{e}^3),$$
$$q_3 = q_2 + a(\mathbf{e}_3, -\mathbf{e}^2) = (\mathbf{e}_1 + a\mathbf{e}_3, \mathbf{e}^1 - a\mathbf{e}^2),$$

for some $a \neq 0$.

In a similar fashion, we see that any point $q_5 \in Q^{\text{dan}}$ that may be joined horizontally to q_1 must be of the form

$$q_5 = q_1 + \left(x\mathbf{e}_1 + y\mathbf{e}_2 - x\mathbf{e}_3, -x\mathbf{e}^1 + x\mathbf{e}^2 + (y - x)\mathbf{e}^3 \right),$$

for some choice of $x, y \in \mathbb{R}$. Likewise, any point $q_4 \in Q^{\text{dan}}$ that may be joined horizontally to q_3 must be of the form

$$q_4 = q_3 + (ab\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3, -ab\mathbf{e}^1 + (a^2 b - c)\mathbf{e}^2 + b\mathbf{e}^3),$$

for some $b, c \in \mathbb{R}$.

Assuming that $q_4 q_5$ is a horizontal line, the horizontality equation $\mathbf{b}_5 - \mathbf{b}_4 = \mathbf{A}_4 \times \mathbf{A}_5$ gives the following system

$$
\begin{cases}
(a + x)b + (1 + y)c &= x - a(1 + y) \\
a(a + x)b + xc &= ax \\
b\,[x - a(1 + y)] &= x
\end{cases}
.
$$

Solving for b, c in terms of x, y in the first two equations and then using the third, implies that:

- if $x \neq 0 \Rightarrow a = 0$, i.e., $q_3 = q_2$, a contradiction;
- if $x = 0 \Rightarrow b = 0 \Rightarrow$ either $c = -a$ (which implies $q_4 = q_2$) or $y = -1$ (which implies $q_5 = q_2$).

This concludes the proof of (3). $\qquad\square$

Appendix A. G_2-symmetry

The definition of the Cartan-Engel distribution using split octonions $\mathbb{O}$ (see Section 2.3) gives rise to an action of the automorphism group of this algebra, $G_2 := \mathrm{Aut}(\mathbb{O})$, as the (local) symmetry group of the various models of this distribution that appear in this article. In this appendix we collect some explicit formulas for the associated infinitesimal action. Most of the material here appeared before, e.g., in [3–5].

A.1. $\mathfrak{g}_2$. The Lie algebra of G_2 is $\mathfrak{g}_2 \subset \mathfrak{so}_{4,3} \subset \mathfrak{so}_{4,4}$, where $\mathfrak{so}_{4,4}$ is the Lie algebra of the subgroup of $\mathrm{GL}(\mathbb{O})$ preserving the inner product on $\mathbb{O}$ of Equation (7), and $\mathfrak{so}_{4,3}$ is the subalgebra preserving the splitting $\mathbb{O} = \mathrm{Re}(\mathbb{O}) \oplus \mathrm{Im}(\mathbb{O})$.

Now $\mathfrak{g}_2$ is the algebra of *derivations* of $\mathbb{O}$, i.e., maps $D \in \mathrm{End}(\mathbb{O})$ satisfying

$$
D(\zeta\zeta') = (D\zeta)\zeta' + \zeta(D\zeta')
$$

for all $\zeta, \zeta' \in \mathbb{O}$. It preserves the decomposition $\mathbb{O} = \mathrm{Re}(\mathbb{O}) \oplus \mathrm{Im}(\mathbb{O})$ and acts trivially on the first summand. On the second summand we have the infinitesimal action of $\mathfrak{g}_2$, defined as follows.

Consider the map $\rho : \mathfrak{sl}_3(\mathbb{R}) \times \mathbb{R}^3 \times (\mathbb{R}^3)^* \to \mathrm{End}(\mathrm{Im}(\mathbb{O}))$,

$$
\rho(T, \mathbf{Q}, \mathbf{p}) : \begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{pA} + \mathbf{bQ} & T\mathbf{A} - \mathbf{p} \times \mathbf{b} + 2\mathbf{Q}x \\ \mathbf{Q} \times \mathbf{A} - \mathbf{b}T + 2\mathbf{p}x & -\mathbf{pA} - \mathbf{bQ} \end{pmatrix}, \quad (25)
$$

where $x \in \mathbb{R}$, $\mathbf{A}, \mathbf{Q} \in \mathbb{R}^3$ (column vectors), $\mathbf{b}, \mathbf{p} \in (\mathbb{R}^3)^*$ (row vectors) and $T \in \mathfrak{sl}_3(\mathbb{R})$ (traceless 3×3 real matrices).

Now ρ is clearly an injective linear map, hence its image is a 14-dimensional subspace of $\mathrm{End}(\mathrm{Im}(\mathbb{O}))$. One can check that $[\rho(T_1, \mathbf{Q}_1, \mathbf{p}_1), \rho(T_2, \mathbf{Q}_2, \mathbf{p}_2)] = \rho(T_3, \mathbf{Q}_3, \mathbf{p}_3)$, where

$$
\begin{aligned}
T_3 &= [T_1, T_2] + 3\,(\mathbf{Q}_1\mathbf{p}_2 - \mathbf{Q}_2\mathbf{p}_1) + (\mathbf{p}_1\mathbf{Q}_2 - \mathbf{p}_2\mathbf{Q}_1)\,\mathrm{I}_3, \\
\mathbf{Q}_3 &= T_1\mathbf{Q}_2 - T_2\mathbf{Q}_1 - 2\mathbf{p}_1 \times \mathbf{p}_2, \\
\mathbf{p}_3 &= \mathbf{p}_1 T_2 - \mathbf{p}_2 T_1 + 2\mathbf{Q}_1 \times \mathbf{Q}_2.
\end{aligned}
$$

It follows that the image of ρ is a 14-dimensional Lie subalgebra of $\mathrm{End}(\mathrm{Im}(\mathbb{O}))$.

Theorem A.1. *The image of ρ in $\mathrm{End}(\mathrm{Im}(\mathbb{O}))$, extended to $\mathbb{O} = \mathrm{Re}(\mathbb{O}) \oplus \mathrm{Im}(\mathbb{O})$ by the 0 action on $\mathrm{Re}(\mathbb{O})$, is the Lie algebra $\mathfrak{g}_2 \subset \mathrm{End}(\mathbb{O})$ of derivations of $\mathbb{O}$.*

See for example [11, page 143].

A.2. The $\mathfrak{g}_2$-action on $\widetilde{Q}^{\mathrm{roll}}$ and Q^{dan}. The G_2-action on $\mathrm{Im}(\mathbb{O})$, given (infinitesimally) by Equation (25), induces actions on $\widetilde{Q}^{\mathrm{oct}}$, Q^{oct}, and a local action on Q^{dan}, then via diagram (8) an action on $\widetilde{Q}^{\mathrm{roll}}$. The latter action does *not* descend to Q^{roll}, even infinitesimally (see next subsection). The next two propositions give explicit formulas for the infinitesimal actions on $Q^{\mathrm{dan}}, \widetilde{Q}^{\mathrm{roll}}$. We start with Q^{dan}.

Proposition A.2. *For each $(T, \mathbf{Q}, \mathbf{p}) \in \mathfrak{sl}_3(\mathbb{R}) \times \mathbb{R}^3 \times (\mathbb{R}^3)^*$, the infinitesimal action of $\rho(T, \mathbf{Q}, \mathbf{p}) \in \mathfrak{g}_2$ on $Q^{\mathrm{dan}} \subset \mathbb{R}^{3,3}$ is given by the vector field $f\partial_{\mathbf{A}} + g\partial_{\mathbf{b}}$ on $\mathbb{R}^{3,3}$, where*

$$
\begin{aligned}
f(\mathbf{A}, \mathbf{b}) &= 2\mathbf{Q} + T\mathbf{A} - \mathbf{p} \times \mathbf{b} - (\mathbf{b}\mathbf{Q} + \mathbf{p}\mathbf{A})\mathbf{A}, \\
g(\mathbf{A}, \mathbf{b}) &= 2\mathbf{p} - \mathbf{b}T + \mathbf{Q} \times \mathbf{A} - (\mathbf{b}\mathbf{Q} + \mathbf{p}\mathbf{A})\mathbf{b}.
\end{aligned}
$$

Proof. This appeared in [4], with somewhat different notation, so we will give it here again for completeness.

We consider the coordinates (x, q) on $\mathrm{Im}(\mathbb{O})$, where $q = (\mathbf{A}, \mathbf{b}) \in \mathbb{R}^{3,3}$. A linear vector field on $\mathrm{Im}(\mathbb{O})$, such as the one given by Equation (25), can be written as

$$
(ax + bq)\partial_x + (cx + dq)\partial_q,
$$

where $a \in \mathbb{R}, b \in (\mathbb{R}^{3,3})^*, c \in \mathbb{R}^{3,3}, d \in \mathrm{End}(\mathbb{R}^{3,3})$. It induces a vector field on $\mathbb{P}(\mathrm{Im}(\mathbb{O}))$, given in the affine chart $\mathbb{R}^{3,3} \to \mathbb{P}(\mathrm{Im}(\mathbb{O})), q \to [1, q]$, by

$$
[c + (d - a)u - (bq)q]\partial_q. \tag{26}
$$

For the linear vector field given by Equation (25), one has

$$
\begin{aligned}
a &= 0, & b(\mathbf{A}, \mathbf{b}) &= \mathbf{p}\mathbf{A} + \mathbf{b}\mathbf{Q}, \\
c &= 2(\mathbf{Q}, \mathbf{p}), & d(\mathbf{A}, \mathbf{b}) &= (T\mathbf{A} - \mathbf{p} \times \mathbf{b}, -\mathbf{b}T + \mathbf{Q} \times \mathbf{A}).
\end{aligned}
$$

Using these in Equation (26) gives the stated formulas. $\qquad \square$

The formulas for the $\mathfrak{g}_2$-action on $\widetilde{Q}^{\mathrm{roll}}$ are more complicated. We shall treat a special representative case with enough detail so that the interested reader can easily derive the formulas of the general case.

Proposition A.3. *Let X^{oct} be the linear vector field on $\mathrm{Im}(\mathbb{O})$ given by formula (25) with $\mathbf{Q} = \mathbf{p} = 0$ and $T = T^t$. The induced infinitesimal action on $\widetilde{Q}^{\mathrm{roll}} = S^2 \times S^3 \subset \mathrm{Im}(\mathbb{H}) \times \mathbb{H}$ is given by the vector field*

$$
X^{\mathrm{roll}} = f\partial_{\mathbf{v}} + g\partial_{\mathbf{w}} + h\partial_s,
$$

where $\mathbf{v} \in \text{Im}(\mathbb{H})$, $s + \mathbf{w} \in \mathbb{H}$ *are the standard Euclidean coordinates on* $\text{Im}(\mathbb{H}) \times \mathbb{H}$, *and*

$$f = T(s\mathbf{v} + \mathbf{v} \times \mathbf{w}) - [T(s\mathbf{v} + \mathbf{v} \times \mathbf{w}) \cdot \mathbf{v}]\,\mathbf{v},$$

$$g = \left[(s^2 - 1)\mathbf{v} + s(\mathbf{v} \times \mathbf{w})\right] \times T\mathbf{v} + (s\mathbf{v} + \mathbf{v} \times \mathbf{w}) \times T(\mathbf{v} \times \mathbf{w})$$
$$+ (\mathbf{v} \cdot \mathbf{w})T(s\mathbf{v} + \mathbf{v} \times \mathbf{w}) - 2\,(T(s\mathbf{v} + \mathbf{v} \times \mathbf{w}) \cdot \mathbf{v})\,\mathbf{w},$$

$$h = T\,(\mathbf{v} + s(\mathbf{v} \times \mathbf{w})) \cdot \mathbf{v} - sT(s\mathbf{v} + \mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} + T(\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w}).$$

Proof. The linear vector field X^{oct} on $\text{Im}(\mathbb{O})$ given by Equation (25) induces, by 'spherization' ($\mathbb{R}^+$-quotient), a vector field on $\widetilde{Q}^{\text{oct}}$ and a vector field X^{roll} on $\widetilde{Q}^{\text{roll}}$ via the diffeomorphism $\Phi : \widetilde{Q}^{\text{roll}} \to \widetilde{Q}^{\text{oct}}$ of diagram (8). Now Φ factors as a composition $\widetilde{Q}^{\text{roll}} \to C \setminus 0 \xrightarrow{\pi} \widetilde{Q}^{\text{oct}}$ (recall that C is the null cone in $\text{Im}(\mathbb{O})$; see Equation (16)). The map $\widetilde{Q}^{\text{roll}} \to C \setminus 0$ is the restriction to $S^2 \times S^3$ of the map $\varphi : \text{Im}(\mathbb{H}) \times \mathbb{H} \to \text{Im}(\mathbb{O})$ given by Equations (16)-(17), $(\mathbf{v}, s + \mathbf{w}) \mapsto \begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{pmatrix}$, where

$$\mathbf{A} = (1 + s)\mathbf{v} + \mathbf{v} \times \mathbf{w},$$
$$\mathbf{b} = (1 - s)\mathbf{v} - \mathbf{v} \times \mathbf{w}, \tag{27}$$
$$x = -\mathbf{v} \cdot \mathbf{w}.$$

The inverse map is given by $\psi : \text{Im}(\mathbb{O}) \to \text{Im}(\mathbb{H}) \times \mathbb{H}$,

$$\mathbf{v} = \frac{\mathbf{A} + \mathbf{b}}{2},$$
$$\mathbf{w} = \frac{\mathbf{A} \times \mathbf{b}}{2} - x\mathbf{v}, \tag{28}$$
$$s = \frac{1}{4}(|\mathbf{A}|^2 - |\mathbf{b}|^2),$$

with derivative

$$\psi^*\,(\mathrm{d}\mathbf{v}) = \frac{1}{2}\,(\mathrm{d}\mathbf{A} + \mathrm{d}\mathbf{b}),$$

$$\psi^*(\mathrm{d}\mathbf{w}) = \frac{1}{2}\,[\mathbf{A} \times \mathrm{d}\mathbf{b} - \mathbf{b} \times \mathrm{d}\mathbf{A} - x(\mathrm{d}\mathbf{A} + \mathrm{d}\mathbf{b}) - (\mathbf{A} + \mathbf{b})\mathrm{d}x],$$

$$\psi^*(\mathrm{d}s) = \frac{1}{2}\,(\mathbf{A} \cdot \mathrm{d}\mathbf{A} - \mathbf{b} \cdot \mathrm{d}\mathbf{b}).$$

That is, φ and ψ, when restricted to $S^2 \times S^3$ and $M := \varphi(S^2 \times S^3) \subset C$ (respectively), are inverse maps:

$$
\begin{array}{ccc}
\text{Im}(\mathbb{H}) \times \mathbb{H} & \begin{array}{c} \xrightarrow{\varphi} \\[-4pt] \xleftarrow[\psi]{} \end{array} & \text{Im}(\mathbb{O}) \\[8pt]
\cup & & \cup \\[8pt]
S^2 \times S^3 & \cong & M
\end{array}
$$

Next, let

$$X^{\mathrm{oct}} = \alpha\partial_{\mathbf{A}} + \beta\partial_{\mathbf{b}} + \gamma\partial_x,$$

and let

$$E := \mathbf{A}\partial_{\mathbf{A}} + \mathbf{b}\partial_{\mathbf{b}} + x\partial_x \tag{29}$$

be the Euler field on $\mathrm{Im}(\mathbb{O})$, which is tangent to C. At each point of $M \subset C$, since X^{oct} is tangent to C, one can decompose uniquely

$$X^{\mathrm{oct}} = X^{\mathrm{oct}}_{\|} + \lambda E,$$

where $X^{\mathrm{oct}}_{\|}$ is tangent M and $\lambda \in \mathbb{R}$. Then $X^{\mathrm{roll}} = \psi_*(X^{\mathrm{oct}}_{\|})$.

Next, we note that by Equations (27), M is given in C by the equation $|\mathbf{A} + \mathbf{b}|^2 = 4$, so TM is the kernel of $(\mathbf{A} + \mathbf{b}) \cdot (\mathrm{d}\mathbf{A} + \mathrm{d}\mathbf{b})$ restricted to TC. It follows that

$$0 = (\mathbf{A} + \mathbf{b}) \cdot (\mathrm{d}\mathbf{A} + \mathrm{d}\mathbf{b})(X^{\mathrm{oct}} - \lambda E) = (\mathbf{A} + \mathbf{b}) \cdot (\alpha + \beta - \lambda(\mathbf{A} + \mathbf{b}))$$
$$= (\mathbf{A} + \mathbf{b}) \cdot (\alpha + \beta) - 4\lambda,$$

thus

$$\lambda = \frac{1}{4}(\mathbf{A} + \mathbf{b}) \cdot (\alpha + \beta) = \frac{\mathbf{v}}{2} \cdot (\alpha + \beta).$$

It follows that

$$f = \mathrm{d}\mathbf{v}(X^{\mathrm{roll}}) = \mathrm{d}\mathbf{v}\left(\psi_*X^{\mathrm{oct}}_{\|}\right) = (\psi^*\mathrm{d}\mathbf{v})X^{\mathrm{oct}}_{\|}$$
$$= \frac{1}{2}(\mathrm{d}\mathbf{A} + \mathrm{d}\mathbf{b})\left(X^{\mathrm{oct}} - \lambda E\right) = \frac{1}{2}\left[\alpha + \beta - ((\alpha + \beta) \cdot \mathbf{v})\mathbf{v}\right],$$

and similarly

$$g = (\psi^*\mathrm{d}\mathbf{w})X^{\mathrm{oct}}_{\|}$$
$$= \left[\frac{1}{2}(\mathbf{A} \times \mathrm{d}\mathbf{b} - \mathbf{b} \times \mathrm{d}\mathbf{A}) - x\mathrm{d}\mathbf{v} - \mathbf{v}\mathrm{d}x\right](X^{\mathrm{oct}} - \lambda E)$$
$$= \frac{1}{2}(\mathbf{A} \times \beta - \mathbf{b} \times \alpha) - \frac{x}{2}\left[\alpha + \beta - ((\alpha + \beta) \cdot \mathbf{v})\mathbf{v}\right]$$
$$\quad - \left(\frac{\mathbf{v}}{2} \cdot (\alpha + \beta)\right)\left[\mathbf{A} \times \mathbf{b} - x(\mathbf{A} + \mathbf{b})\right] - \gamma\mathbf{v},$$
$$h = (\psi^*\mathrm{d}s)X^{\mathrm{oct}}_{\|} = \frac{1}{2}(\mathbf{A} \cdot \mathrm{d}\mathbf{A} - \mathbf{b} \cdot \mathrm{d}\mathbf{b})\left(X^{\mathrm{oct}} - \lambda E\right)$$
$$= \frac{1}{2}\left[\mathbf{A} \cdot \alpha - \mathbf{b} \cdot \beta - (\mathbf{v} \cdot (\alpha + \beta))\left(|\mathbf{A}|^2 - |\mathbf{b}|^2\right)\right].$$

Now we apply the above to the vector field X^{oct} on $\mathrm{Im}(\mathbb{O})$ given by Equation (25), with $T = T^t$, $\mathbf{Q} = 0$, $\mathbf{p} = 0$. Then

$$\alpha = T\mathbf{A}, \ \beta = -T\mathbf{b}, \ \gamma = 0.$$

Using these values in the above expressions for f, g, h, we obtain, after some simplification, the stated formulas. $\qquad\square$

A.3. The $\mathfrak{g}_2$-action on $\widetilde{Q}^{\mathrm{roll}}$ does not descend to Q^{roll}. It was shown in [5, §7] that the G_2-action on $\widetilde{Q}^{\mathrm{roll}}$ does not descend to Q^{roll}. Here we show that the infinitesimal $\mathfrak{g}_2$-action does not descend either (a stronger result). Actually, we will determine the elements of $\mathfrak{g}_2$ whose action on $\widetilde{Q}^{\mathrm{roll}}$ descend to Q^{roll}.

Proposition A.4. *Consider the* $2 : 1$ *cover* $\widetilde{Q}^{\mathrm{roll}} \to Q^{\mathrm{roll}}$ *and the vector field* X^{roll} *on* $\widetilde{Q}^{\mathrm{roll}}$ *induced by an element* $\rho(T, \mathbf{Q}, \mathbf{p}) \in \mathfrak{g}_2$, *see Equation* (25). *Then* X^{roll} *descends to a vector field on* Q^{roll} *if and only if* $T^t = -T$, $\mathbf{Q} = -\mathbf{p}$.

Proof. Recall that $\widetilde{Q}^{\mathrm{roll}} = S^2 \times S^3$, $Q^{\mathrm{roll}} = S^2 \times \mathrm{SO}_3$ and $\widetilde{Q}^{\mathrm{roll}} \to Q^{\mathrm{roll}}$ is the quotient by the antipodal map in the second factor. Let us denote this self-map of $\widetilde{Q}^{\mathrm{roll}}$ by σ. The vector field X^{roll} on $\widetilde{Q}^{\mathrm{roll}}$ thus descends to Q^{roll} if and only if it is σ-invariant, $\sigma_* X^{\mathrm{roll}} = X^{\mathrm{roll}}$.

The diffeomorphism $\Phi : \widetilde{Q}^{\mathrm{roll}} \to \widetilde{Q}^{\mathrm{oct}}$ maps X^{roll} to a vector field $\Phi_* X^{\mathrm{roll}}$ on $\widetilde{Q}^{\mathrm{oct}}$. Thus X^{roll} is σ-invariant if and only if $\Phi_* X^{\mathrm{roll}}$ is τ-invariant, where $\tau = \Phi \circ \sigma \circ \Phi^{-1}$.

Using the definition of Φ (Equation (9)), one finds that τ is given by a linear involution of $\mathrm{Im}(\mathbb{O})$,

$$\widetilde{\tau} : \begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{pmatrix} \mapsto \begin{pmatrix} -x & \mathbf{b} \\ \mathbf{A} & x \end{pmatrix}$$

Namely, if $\zeta \in C \setminus 0$ (a non-zero null octonion) then $\tau([\zeta]) = [\widetilde{\tau}\zeta]$.

Similarly, the vector field $\Phi_* X^{\mathrm{roll}}$ on $\widetilde{Q}^{\mathrm{oct}}$ is given by the linear vector field X^{oct} on $\mathrm{Im}(\mathbb{O})$ of Equation (25) by first restricting X^{oct} to the null cone $C \subset \mathrm{Im}(\mathbb{O})$ (this restriction makes sense since X^{oct} is tangent to $C \setminus 0$), then projecting via the $\mathbb{R}^+$-quotient $\pi : C \setminus 0 \to \widetilde{Q}^{\mathrm{oct}}$.

Summarizing, the σ-invariance of X^{roll} amounts to

$$X^{\mathrm{oct}}(\widetilde{\tau}\zeta) \equiv \widetilde{\tau}_* \left(X^{\mathrm{oct}}(\zeta) \right) \bmod E(\widetilde{\tau}\zeta), \ \forall \zeta \in C, \tag{30}$$

where E is the Euler vector field on $\mathrm{Im}(\mathbb{O})$ (see Equation (29)).

Next, recall from (25) that if $\zeta = \begin{pmatrix} x & \mathbf{A} \\ \mathbf{b} & -x \end{pmatrix} \in \mathrm{Im}(\mathbb{O})$ then

$$X^{\mathrm{oct}}(\zeta) = (T\mathbf{A} - \mathbf{p} \times \mathbf{b} + 2x\mathbf{Q})\partial_{\mathbf{A}} + (\mathbf{Q} \times \mathbf{A} - T^t\mathbf{b} + 2x\mathbf{p})\partial_{\mathbf{b}}$$
$$+ (\mathbf{p} \cdot \mathbf{A} + \mathbf{b} \cdot \mathbf{Q})\partial_x,$$

hence

$$X^{\mathrm{oct}}(\widetilde{\tau}\zeta) = (T\mathbf{b} - \mathbf{p} \times \mathbf{A} - 2x\mathbf{Q})\partial_{\mathbf{A}} + (\mathbf{Q} \times \mathbf{b} - T^t\mathbf{A} - 2x\mathbf{p})\partial_{\mathbf{b}}$$
$$+ (\mathbf{p} \cdot \mathbf{b} + \mathbf{A} \cdot \mathbf{Q})\partial_x$$

and

$$\widetilde{\tau}_* \left(X^{\mathrm{oct}}(\zeta) \right) = (\mathbf{Q} \times \mathbf{A} - T^t\mathbf{b} + 2x\mathbf{p})\partial_{\mathbf{A}} + (T\mathbf{A} - \mathbf{p} \times \mathbf{b} + 2x\mathbf{Q})\partial_{\mathbf{b}}$$
$$- (\mathbf{p} \cdot \mathbf{A} + \mathbf{b} \cdot \mathbf{Q})\partial_x.$$

Hence, Equation (30) is equivalent to

$$
\begin{pmatrix} \mathbf{Q} \times \mathbf{A} - T^t \mathbf{b} + 2x\mathbf{p} \\ T\mathbf{A} - \mathbf{p} \times \mathbf{b} + 2x\mathbf{Q} \\ -\mathbf{p} \cdot \mathbf{A} - \mathbf{b} \cdot \mathbf{Q} \end{pmatrix} \equiv \begin{pmatrix} T\mathbf{b} - \mathbf{p} \times \mathbf{A} - 2x\mathbf{Q} \\ \mathbf{Q} \times \mathbf{b} - T^t \mathbf{A} - 2x\mathbf{p} \\ \mathbf{p} \cdot \mathbf{b} + \mathbf{A} \cdot \mathbf{Q} \end{pmatrix} \bmod \begin{pmatrix} \mathbf{b} \\ \mathbf{A} \\ -x \end{pmatrix}
$$

for all $\mathbf{A}, \mathbf{b}, x$ such that $\mathbf{b}\mathbf{A} = x^2$. Setting $\mathbf{b} = 0$, $x = 0$ in the above equation, the third component reads $(\mathbf{Q} + \mathbf{p}) \cdot \mathbf{A} = 0$ for all $\mathbf{A}$, hence $\mathbf{Q} = -\mathbf{p}$. The second component then reads $(T + T^t)\mathbf{A} = \lambda \mathbf{A}$ for all $\mathbf{A}$ for some λ (that may depend on $\mathbf{A}$). Hence, $T + T^t$ is a multiple of the identity. But T is traceless, and so is $T + T^t$, hence $T + T^t = 0$, as needed. $\qquad\square$

Remark A.5. The subspace of $\mathfrak{g}_2$ indicated in this proposition is isomorphic to $\mathfrak{so}_3 \oplus \mathfrak{so}_3 = \mathfrak{so}_4$, the Lie algebra of a maximal compact subgroup $SO_4 \simeq K \subset G_2$, which has an 'obvious' action on $Q^{\mathrm{roll}} = S^2 \times SO_3$. In fact, the SO_4-action descends to $SO_3 \times SO_3$ (a $\mathbb{Z}_2$-quotient of SO_4), $(g_1, g_2) : (\mathbf{v}, g) \mapsto (g_1 \mathbf{v}, g_1 g g_2^{-1})$. The embedding $SO_4 \to G_2$, and the associated embedding $\mathfrak{so}_3 \oplus \mathfrak{so}_3 \to \mathfrak{g}_2$, has appeared explicitly above during the proof of Theorem 2.12; see Section 3.3, item (3), step (c), Equation (20). This also appeared in [5], §2.2 and §5.

Appendix B. Coordinate formulae for $\mathscr{D}^{\mathrm{roll}}$

The configuration space Q^{roll} of rolling a sphere of radius $1/\rho$ on a stationary sphere of radius 1 is $S^2 \times SO_3$, where $\mathbf{v} \in S^2$ denotes the contact point of the two spheres and $g \in SO_3$ the orientation of the moving sphere with respect to some fixed initial orientation.

To write down the rolling distribution on Q^{roll}, we will use the coordinates $(\phi, \theta, \alpha, \beta, \gamma)$, where ϕ, θ are the spherical coordinates on S^2,

$$
\mathbf{v} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}
$$

and α, β, γ are Euler angles on SO_3,

$$
g = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}
$$

The angular velocity vector of the moving sphere, about its center, is

$$
\omega = \begin{pmatrix} \dot\alpha \sin\beta \sin\gamma + \dot\beta \cos\gamma \\ \dot\alpha \sin\beta \cos\gamma - \dot\beta \sin\gamma \\ \dot\alpha \cos\beta + \dot\gamma \end{pmatrix}
$$

The rolling distribution $\mathscr{D}^{\mathrm{roll}}$ on Q^{roll} is the rank 2 distribution whose integral curves satisfy $(1 + \rho)\dot{\mathbf{v}} = \omega \times \mathbf{v}$, $\omega \cdot \mathbf{v} = 0$ (see Definition 2.6). Explicitly,

$$(1 + \rho)(\dot{\theta} \cos \theta \cos \phi - \dot{\phi} \sin \theta \sin \phi) - (\dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma) \sin \theta \cos \phi$$
$$+ (\dot{\alpha} \sin \beta \cos \gamma - \dot{\beta} \sin \gamma) \sin \theta \sin \phi - \dot{\alpha} \cos \beta \cos \theta = 0,$$

$$(1 + \rho)(\dot{\theta} \cos \theta \cos \phi - \dot{\phi} \sin \theta \sin \phi) - (\dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma) \sin \theta \cos \phi$$
$$+ (\dot{\alpha} \sin \beta \cos \gamma - \dot{\beta} \sin \gamma) \sin \theta \sin \phi - \dot{\alpha} \cos \beta \cos \theta = 0,$$

$$(1 + \rho)(-\dot{\theta} \sin \theta) + (\dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma) \cos \theta + \dot{\alpha} \sin \beta \cos \theta = 0,$$

$$\sin \theta \cos \phi (\dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma) + \sin \theta \sin \phi (\dot{\alpha} \sin \beta \cos \gamma - \dot{\beta} \sin \gamma)$$
$$+ \cos \theta (\dot{\alpha} \cos \beta + \dot{\gamma}) = 0.$$

Remark B.1. The first three equations are linearly dependent, defining a rank 3 distribution on Q^{roll}. The relation is $\mathbf{v} \cdot [(1 + \rho)\dot{\mathbf{v}} - \boldsymbol{\omega} \times \mathbf{v}] = 0$. Thus, one can omit say the first equation, away from the subset of Q^{roll} where $v_1 = 0$.

References

[1] AGRACHEV, ANDREI A. Rolling balls and octonions. *Proc. Steklov Inst. Math.* **258** (2007), no. 1, 13–22. ISBN:978-5-02-036672-5, ISBN:978-5-02-035888-1. MR2400520, Zbl 1153.70010, arXiv:math/0611812, doi: 10.1134/S0081543807030030. 985

[2] AN, DANIEL; NUROWSKI, PAWEŁ. Twistor space for rolling bodies. *Comm. Math. Phys.* **326** (2014), no. 2, 393–414. MR3165459, Zbl 1296.53100, arXiv:1210.3536, doi: 10.1007/s00220-013-1839-2. 985

[3] BAEZ, JOHN C.; HUERTA, JOHN. G_2 and the rolling ball. *Trans. Amer. Math. Soc.* **366** (2014), no. 10, 5257–5293. MR3240924, Zbl 1354.20031, arXiv:1205.2447, doi: 10.1090/S0002-9947-2014-05977-1. 982, 985, 991, 994, 1008

[4] BOR, GIL; HERNÁNDEZ-LAMONEDA, LUIS; NUROWSKI, PAWEŁ. The dancing metric, G_2-symmetry and projective rolling. *Trans. Amer. Math. Soc.* **370** (2018), no. 6, 4433–4481. MR3811534, Zbl 1387.53021, arXiv:1506.00104, doi: 10.1090/tran/7277. 982, 983, 989, 990, 992, 1000, 1008, 1009

[5] BOR, GIL; MONTGOMERY, RICHARD. G_2 and the rolling distribution. *Enseign. Math.* (2) **55** (2009), no. 1-2, 157–196. MR2541507, Zbl 1251.70008, arXiv:math/0612469, doi: 10.4171/LEM/55-1-8. 982, 985, 989, 991, 1008, 1012, 1013

[6] BRYANT, ROBERT L. Élie Cartan and geometric duality. *Journées Élie Cartan 1998 et 1999* **16** (2000), 5–20. https://scholars.duke.edu/display/pub1139888. 989

[7] BRYANT, ROBERT L.; HSU, LUCAS. Rigidity of integral curves of rank 2 distributions. *Invent. Math.* **114** (1993), no. 2, 435–461. MR1240644, Zbl 0807.58007, doi: 10.1007/BF01232676. 982, 985, 989, 990, 991

[8] CARTAN, ÉLIE. Sur la structure des groupes finis et continus. *C. R. Math. Acad. Sci. Paris* **116** (1893), 962–964. JFM 25,0637.02. 989

[9] CARTAN, ÉLIE. Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Sci. École Norm. Sup. (3)* **27** (1910), 109–192. MR1509120, JFM 41.0417.01. 989

[10] ENGEL, FRIEDRICH. Sur un groupe simple à quatorze paramètres. *C. R. Math. Acad. Sci. Paris* **116** (1893), 786–788. JFM 25.0631.01. 989

[11] JACOBSON, NATHAN. Lie Algebras. *Dover Publications, Inc., New York*, 1979. ix+331 pp. ISBN: 0-486-63832-4. MR0559927. 992, 1009

[12] KAPLANSKY, IRVING. Linear algebra and geometry. A second course. *Allyn and Bacon, Inc.,* Boston, Mass., 1969. xii+139 pp. MR0249444, Zbl 0184.24201. 996

[13] MURPHY, BERNARD. Rolling circles. An animation using *GeoGebra*. https://www.geogebra.org/m/sehp2VmH. 986

[14] SAGERSCHNIG, KATJA. Split octonions and generic rank two distributions in dimension five. *Arch. Math. (Brno)* **42** (2006), 329–339. MR2322419, Zbl 1164.53362. 992

[15] Spherical geometry. *Wikipedia*. https://en.wikipedia.org/wiki/Spherical_geometry. 999

[16] ZORN, MAX A. Theorie der alternativen ringe. *Abh. Math. Sem. Univ. Hamburg* **8** (1931), no. 1, 123–147. MR3069547, Zbl 56.0140.01, doi: 10.1007/BF02940993. 992

(Gil Bor) CIMAT, A.P. 402, GUANAJUATO, GTO. 36000, MEXICO
gil@cimat.mx

(Luis Hernández Lamoneda) CIMAT, A.P. 402, GUANAJUATO, GTO. 36000, MEXICO
lamoneda@cimat.mx

This paper is available via http://nyjm.albany.edu/j/2023/29-38.html.